Lecture 2

Matrices (Review)

1) A matrix, A, is a collection of numbers, variables, elements a_{ij} $i = 1, \dots m$ and $j = 1, \dots n$ arranged in a rectangular array within m rows and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{m \times n}$$

2) If $m = n$ then A is a square matrix

3) for a square matrix $\sum_{i=1}^{n} a_{ii} \triangleq$ (is defined as) trace of the matrix

Egs: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}$ Column vector $\qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ square matrix

trace $= 1 + 4 = 5$

$$\begin{bmatrix} \frac{2}{s+4} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{4}{s+5} \\ \frac{5}{s+6} & \frac{6}{s+7} \end{bmatrix}_{3\times 2}$$ (3x2) rational matrix

$\begin{bmatrix} x_1 & x_2 & x_3^2 \end{bmatrix}_{1\times 3}$ row vector

4) Diagonal matrix : A square matrix whose off diagonal elements (a_{ij} $i \neq j$) are 0

5) Identity matrix: A diagonal matrix whose diagram elements are 1

Matrix operations

6) Addition and Substraction of matrices $A_{m_A \times n_A}$, $B_{m_B \times n_B}$ is defined if and only if $m_A = m_B$ and $n_A = n_B$

if this Condition is satisfied then

$$C = A \pm B \quad \text{with} \quad c_{ij} = a_{ij} \pm b_{ij}$$

Eg: $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 4 & 6 \end{bmatrix}$

7) Multiplication

a) Scalar multiplication

$$2 . \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 8 & 12 \end{bmatrix}$$

b) matrix multiplication of matrices $A_{m_A \times n_A}$, $B_{m_B \times n_B}$ to form product AB is defined if and only if $n_A = m_B$

If this Condition is Satisfied then

$$C_{m_A \times n_B} = AB \Rightarrow C_{ij} = \sum_{k=1}^{m_B} a_{ik} b_{kj}$$

Eg: $A = [1, 2, 3]_{1\times 3}$ $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$

$$C = A.B = 1 + \underset{2\times 2}{\underset{\sim}{4}} + \underset{3\times 3}{\underset{\sim}{9}} = 14$$

$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ $AB = \begin{bmatrix} 1.2 + 3.3 & 1.1 + 3.5 \\ 2.2 + 4.3 & 2.1 + 4.5 \end{bmatrix}$

$$AB = \begin{bmatrix} 11 & 16 \\ 16 & 22 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 10 \\ 13 & 29 \end{bmatrix}$$

Properties:

$AB \neq BA$

$AB = 0$ does not imply $A = 0$ or $B = 0$

$AB = AC$ does not imply $B = C$

$A(B+C) = AB + AC$

$(A+B)C = AC + BC$

$A(BC) = (AB)C = ABC$

8) Transpose of a matrix A is denoted as A' or A^T and is the matrix obtained by interchanging rows and columns

Eg $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

Proprelies : $(AB)^T = B^T A^T$; $(ABCD)^T = D^T C^T B^T A^T$

11) Determinant of a Square matrix $A_{n \times n}$

$$\det(A) = \sum_j a_{ij} \gamma_{ij} \quad \text{for any } i = 1, \dots n.$$

Where γ_{ij} denotes the Cofactor corresponding to a_{ij} and is equal to $(-1)^{i+j} \det(M_{ij})$. Where the M_{ij} is the $(n-1)(n-1)$ matrix obtained by deleting the ith row and the jth Column of A.

Det M_{ij} is Called the ijth minor.

12) The determinant of a triangular matrix is equal to the product of ~~the elements~~ the diagonal elements.

13) Adjoint of a Square matrix A ~~[Aij]~~ is denoted $[\gamma_{ij}]^T$ where γ_{ij} are the Cofactors defined in 11).

Eg: $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$\gamma_{11} = 3 \qquad \gamma_{12} = -4$

$\gamma_{21} = -2 \qquad \gamma_{22} = 1$

$$\det(A) = a_{11}\gamma_{11} + a_{12}\gamma_{12}$$
$$= 1 \times 3 - 2 \times 4 = -5$$

$$\mathrm{Adj}(A) = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}$$

Eg: triangular matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$$

$A = [a_{ij}]_{m \times m}$ is triangular if $a_{ij} = 0$ for $j > i$
or if $a_{ij} = 0$ for $j < i$

$$\det(A) = 1 \times 2 \times 3 = 6$$

14) Inverse of a square matrix A is denoted A^{-1} and is defined as

$$A^{-1} = \frac{\mathrm{Adj}(A)}{\det(A)}$$

If $\det(A) = 0$ then A^{-1} does not exist and A is said to be singular.

Eg:
$$A = \begin{bmatrix} S+2 & S+3 \\ S+4 & S+5 \end{bmatrix}$$

$$\det(A) = (S+2)(S+5) - (S+4)(S+3)$$
$$= \cancel{S^2} + \cancel{2S} + \cancel{5S} + 10 - \cancel{S^2} - \cancel{4S} - \cancel{3S} - 12 = -2$$

$$\mathrm{Adj}(A) = \begin{bmatrix} S+5 & -(S+3) \\ -(S+4) & S+2 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} S+5 & -(S+3) \\ -(S+4) & (S+2) \end{bmatrix}$$

Properties:

$$A^{-1}A = AA^{-1} = I \quad \text{(Identity matrix)}$$

$AX = B \implies X = A^{-1}B$ provided that A is square and A^{-1} exists.

15) Eigenvalues of a square matrix A is defined as the roots (real / Complex) of the equation

$|\lambda I - A| = 0$ (whe $|\ |$ denotes determinant

e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad \lambda I - A = \begin{bmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{bmatrix}$

$$|\lambda I - A| = (\lambda - 1)(\lambda - 4) - (-2)(-3) = 0$$

$$\lambda^2 - 5\lambda - 2 = 0$$

$$\implies \lambda = \frac{5 \pm \sqrt{25 + 8}}{2}$$

$$\lambda_1 = 5.37 \qquad \lambda_2 = -0.37$$

For a $n \times n$ matrix the will be n eigenvalues.

16) Eigenvectors x_1 of a square matrix associated with an eigenvalue λ_1 is defined as any Column vector X_1 Satisfying

$$A X_1 = \lambda_1 X_1 \implies (\lambda_1 I - A) X_1 = 0$$

Eg: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\lambda_1 = 5.37$ $\lambda_2 = -0.37$

$$\lambda_1 I - A = \begin{pmatrix} 4.37 & -2 \\ -3 & 1.37 \end{pmatrix}$$

let $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

then $\begin{bmatrix} 4.37 & -2 \\ -3 & 1.37 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

We get $x_1 = 0.4574\, x_2$

Choose $x_2 = 1$ then $x_1 = 0.4572$

$$X_1 = \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$$

Similarly for $\lambda_2 = -0.37 \Rightarrow X_2 = \begin{bmatrix} -1.4574 \\ 1 \end{bmatrix}$

let $T = [\cancel{X}_1 \,; X_2] = \begin{bmatrix} 0.4574 & -1.4574 \\ 1 & 1 \end{bmatrix}$

$$T^{-1} = \frac{1}{1.9148} \begin{bmatrix} 1 & 1.4574 \\ -1 & 0.4574 \end{bmatrix} = \begin{bmatrix} 0.5222 & 0.761 \\ -0.5222 & 0.23 \end{bmatrix}$$

$$T^{-1} A T = \begin{bmatrix} 5.37 & 0 \\ 0 & -0.37 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Eg: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\lambda_1 = 5.37$ $\lambda_2 = -0.37$

$$\lambda_1 I - A = \begin{pmatrix} 4.37 & -2 \\ -3 & 1.37 \end{pmatrix}$$

let $x_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

then [illegible]

<u>In general</u> if $\lambda_1, \lambda_2 \ldots \lambda_n$ are n eigenvalues of matrix A of size $n \times n$ & if $x_1, x_2, \ldots x_n$ are n different eigenvectors associated with the corresponding eigenvalues & if

$$T = [x_1 \; x_2 \; \ldots \; x_n]$$

then

$$T^{-1} A T = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Eg: $\begin{bmatrix} \cancel{x_1} & \cancel{x_4} \\ \cancel{x_2} & \cancel{x_5} \\ \cancel{x_3} & \cancel{x_6} \end{bmatrix}$ $\quad A$