CASE 2: we have a zero in the first column and the elements in the corrispond row are not zero.

If only one element in the RH array is zero, it maybe replaced with a small positive number ϵ.

EX:

$$s^5+2s^4+2s^3+4s^2+11s+10=0$$

s^5	1	2	11	0
s^4	2	4	10	0
s^3	0	6	0	
s^2	c_1	c_2		
s^1	d_1			
s^0				

$$b_1 = -\frac{1}{2}\begin{vmatrix}1 & 2\\ 2 & 4\end{vmatrix} = 0$$

$$b_2 = -\frac{1}{2}\begin{vmatrix}1 & 11\\ 2 & 10\end{vmatrix} = -\frac{1}{2}(10-22) = 6$$

$b_3 =$ obviously zero.

for b_1, since all the values in the b row are no zero, we replace b_1 with ϵ

$$c_1 = -\frac{1}{\epsilon}\begin{vmatrix}2 & 4\\ \epsilon & 6\end{vmatrix} = -\frac{1}{\epsilon}(12-4\epsilon) = 4-\frac{12}{\epsilon}$$

$$c_2 = -\frac{1}{\epsilon}\begin{vmatrix}2 & 10\\ \epsilon & 0\end{vmatrix} = -\frac{1}{\epsilon}(-10\epsilon) = 10$$

$$d_1 = \frac{-1}{4-\frac{12}{\epsilon}}\begin{vmatrix}\epsilon & 6\\ 4-\frac{12}{\epsilon} & 10\end{vmatrix} = \frac{-1}{4-\frac{12}{\epsilon}}\left(10\epsilon - 6\left(4-\frac{12}{\epsilon}\right)\right) = 6-\frac{10\epsilon}{4-\frac{12}{\epsilon}}$$

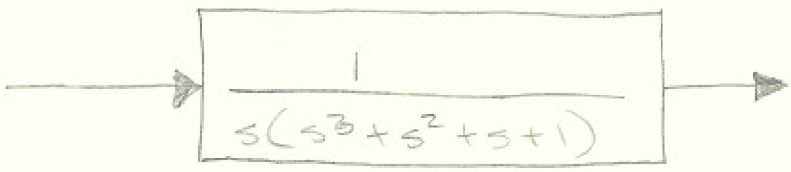
$$e_1 = \frac{-1}{d_1}\begin{vmatrix} d_1 & 10 \\ e_1 & 0 \end{vmatrix} = \frac{-1}{e_1}(-10e_1) = 10$$

∴ the right hand column

$$\begin{array}{c} + \\ + \\ + \\ - \\ + \\ + \end{array}\begin{bmatrix} 1 \\ 2 \\ \epsilon \\ -\infty \\ 6 \\ 10 \end{bmatrix}$$

∴ there are 2 poles in the right hand plane.

Ex: consider the feedback system



? can this be stabilized with just a proportional controller ?

$$G_c(s) = \frac{K}{s(s^3+s^2+s+1) + K}$$

$$\begin{array}{c|cccc} s^4 & 1 & 1 & K & \emptyset \\ s^3 & 1 & 1 & \emptyset & \emptyset \\ s^2 & b_1 & b_2 & \emptyset & \\ s^1 & c_1 & & & \\ s^0 & d_1 & & & \end{array}$$

$$b_1 = -1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$b_2 = -1\begin{vmatrix} 1 & K \\ 1 & \emptyset \end{vmatrix} = -1(-K) = K \qquad \therefore\ b_1 = \epsilon$$

$$c_1 = -\frac{1}{\epsilon}\begin{vmatrix} 1 & 1 \\ \epsilon & K \end{vmatrix} = -\frac{1}{\epsilon}(K - \epsilon) = -\frac{K}{\epsilon} + 1$$

$$d_1 = -\frac{1}{c_1}\begin{vmatrix} \epsilon & K \\ c_1 & 0 \end{vmatrix} = -\frac{1}{c_1}(-c_1 K) = K$$

∴ the right hand column of the RH method is.

1	+
1	+
K	+
$-\infty$	−
K	+

∴ we conclude that no matter what we do there is two poles in the right hand plane.

CASE: zeros in the first column, and the other elements in the same row are also zero.

or the row is a single element that contains a zero.

This situation occurs when the characteristic equ ① has solutions that are symetrically located about the origin of the complex plane (ie - 2 real roots $\bar{w}$ equal magnitude and opposite sign, or/and 2 conjugate iminary roots).

ie $(s-2)(s+2)$ or/and $(s+j\omega)(s-j\omega)$

EX: In such a case, the evaluation of the rest of the array can be continued by forming an auxillary polynomial with the coeffients of the last row, and by using the coeffients of the derivative of this polynomial in the next row.

Such roots with the equal magnetude and lying radially opposite to the complex plane can be found by solving the auxiliary polynomial, which is always even.

EX:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

$$\begin{array}{c|cccc} s^5 & 1 & 24 & -25 & \emptyset \\ s^4 & 2 & 48 & -50 & \emptyset \\ s^3 & b_1 & b_2 & & \\ s^2 & 24 & -50 & & \\ s^1 & 112.7 & & & \\ s^0 & -50 & & & \end{array}$$

$$b_1 = \frac{-1}{2} \begin{vmatrix} 1 & 24 \\ 2 & 48 \end{vmatrix} = \emptyset$$

$$b_2 = \frac{-1}{2} \begin{vmatrix} 1 & -25 \\ 2 & -50 \end{vmatrix} = \emptyset$$

$\therefore$ we have a row of zeros in the "b" area

$$P(s) = 2s^4 + 48s^2 - 50$$

$$\frac{dP}{ds} = 8s^3 + 96s =$$

$\therefore$ $b_1 = 8$

$b_2 = 96$

we observe that the roots of $P(s)$

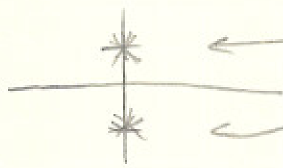
$$P(s) = 2(s+1)(s-1)(s-5j)(s+5j)$$

CASE 4:

repeated roots of the characteristic equ ① on the jω axis.

note: this is the only case when the RH method may give a wrong answer.

If the roots on the jω axis are simple, then the system is neither stable, or unstable, it is marginally unstable. If the jω axis roots are repeated, the system responce will be unstable with the form of $[t\sin(\omega t + \phi)]$



many poles at each point.

the RH method will not reveal this form of instibility, but it is possible to see it from the aux polynomial.

Ex:

$$s^5 + s^4 + 2s^3 + 2s^2 + s + 1 = 0$$

$$\begin{array}{c|ccc}
s^5 & 1 & 2 & 1 \\
s^4 & 1 & 2 & 1 \\
\hline
s^3 & 0 & 0 & 0 \\
s^2 & & & \\
s^1 & & & \\
s^0 & & &
\end{array}$$

∴ we must take aux polynomial.

$$P(s) = s^4 + 2s^2 + 1$$

$$\frac{dP}{ds} = 4s^3 + 4s$$

now we can use

4 4

for the element in the first row

$$\begin{array}{c|ccc}
s^5 & 1 & 2 & 1 \\
s^4 & 1 & 2 & 1 \\
s^3 & 4 & 4 & \emptyset \\
s^2 & 1 & 1 & \emptyset \\
s^1 & \emptyset & \emptyset & \emptyset \\
s^0 & & &
\end{array} \longrightarrow P(s) = s^2 + 1$$

again we get a row of zeros. $\frac{dP}{ds} = 2s$

$$\begin{array}{c|ccc}
s^5 & 1 & 2 & 1 \\
s^4 & 1 & 2 & 1 \\
s^3 & 1 & 1 & \emptyset \\
s^2 & 1 & 1 & \emptyset \\
s^1 & 1 & \emptyset & \emptyset \\
s^0 & 1 & \emptyset & \emptyset
\end{array}$$

we are concerned to know if there are any repeated roots on the imaginary axis.

we know that the first aux polynomial has the same roots as the system.

we notice that the poles are $(s^2+1)^2$

∴ we do endup with repeated roots on the imaginary axis.

note: the repeated roots on the imaginary axis will be visible when the aux polynomial

RELATIVE STABILITY PROCESS/ANALYSIS.

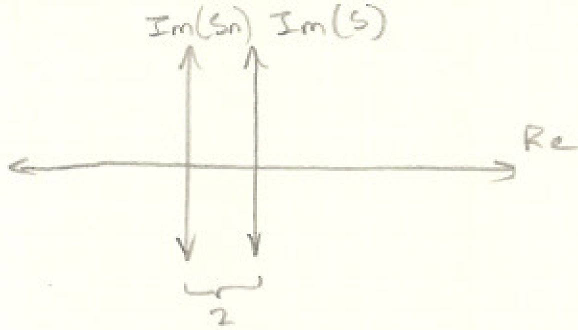
$$q(s) = s^3 + 4s^2 + 6s + 4 = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 6 \\ s^2 & 4 & 4 \\ s & b_1=5 & \emptyset \\ s^0 & b_2=4 & \emptyset \end{array}$$

$$b_1 = -\frac{1}{4}\begin{vmatrix} 1 & 6 \\ 4 & 4 \end{vmatrix} = 5$$

$$b_2 = -\frac{1}{5}\begin{vmatrix} 4 & 4 \\ 5 & \emptyset \end{vmatrix} = 4$$

we can shift the imaginary axis to compare



$$s_n = s + 2$$

$$s = s_n - 2$$

$$q(s_n) = (s_n-2)^3 + 4(s_n-2)^2 + 6(s_n-2) + 4 = 0$$

$$= s_n^3 - 2s_n^2 + 2s_n = \emptyset$$

b/c of the change in coeffient. We know that the are a Pole(s) in the RHP wrt the new imaginary axis.

EX: now we try another shift of the imaginary axis.

$$s_n = s + 1$$

$$s = s_n - 1$$

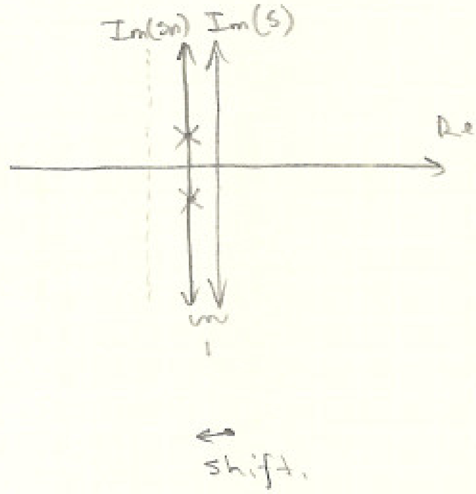
$$q(s_n) = s_n^3 + s_n^2 + s_n + 1$$

s^3	1	1
s^2	1	1
s^1	0	0
s^0	1	

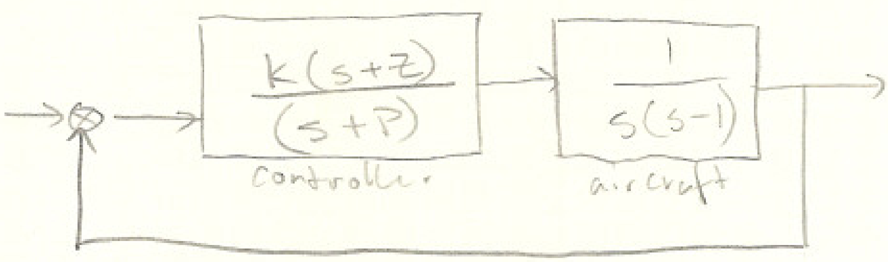
s^2 row $\longrightarrow P(s) = s^2 + 1$

s^1 row $\longleftarrow \boxed{2 \quad 0}$

∴



EX: control system for the orientation of an aircraft.



find k, z, p to make the closed loop system stable.

$$G_c(s) = \frac{N(s)}{D(s)}$$

$$D(s) = s^3 + (p-1)s^2 + (k-p)s + kz = 0$$

$$\begin{array}{c|cc}
s^3 & 1 & k-p \\
s^2 & p-1 & kz \\
s^1 & b_1 & \\
s^0 & b_2 &
\end{array}$$

$$b_1 = \frac{-1}{p-1}\begin{vmatrix} 1 & k-p \\ p-1 & kz \end{vmatrix} = \frac{(k-p)(p-1) - kz}{p-1}$$

$$b_2 = kz.$$

∴ we know that.

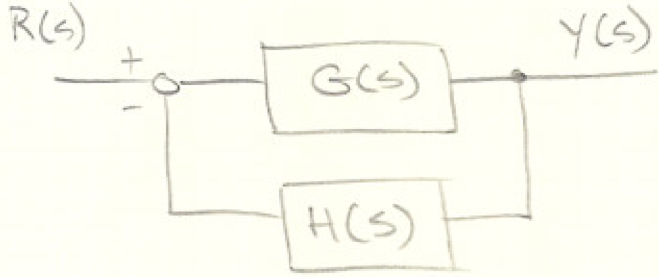
$$p-1 > 0$$

$$k-p > 0$$

$$kz > 0$$

$$\frac{(k-p)(p-1) - kz}{(p-1)} > 0$$

ROOT LOCUS METHOD.



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + H(s)G(s)}$$

then the characteristice equ becomes.

$$1 + G(s)H(s) = 0$$
$$G(s)H(s) = -1$$

angle condition.

$$\angle G(s)H(s) = \pm 180°(2k+1) \quad k \in I$$

magnitude condition

$$|G(s)H(s)| = 1$$

in many cases $G(s)H(s)$ involves a gain of k, and the characteristic equation can be writen as follows.

$$1 + k\left\{\frac{(s+z_1)(s+z_2)\ldots(s+z_n)}{(s+P_1)(s+P_2)\ldots(s+P_m)}\right\} = 0$$

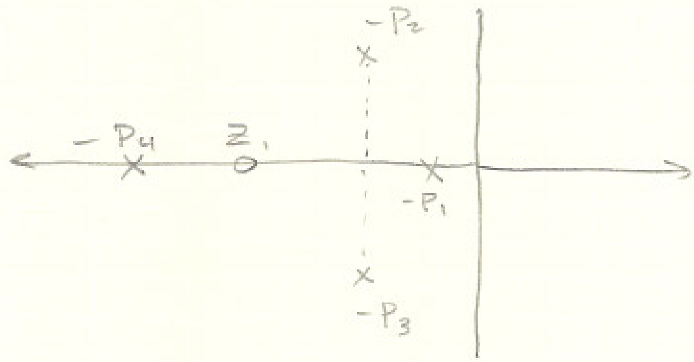
$$(s+P_1)(s+P_2)\ldots(s+P_n) + k(s+z_1)(s+z_2)\ldots(s+z_m) = 0$$

* The root loci for a system are the loci of the closed loop poles as the gain k is varied from 0 to infinity.

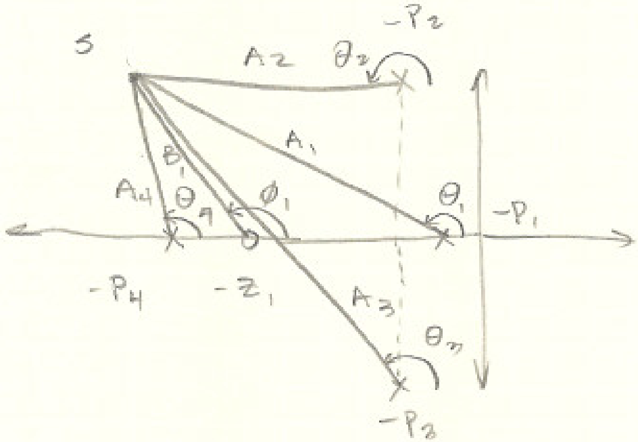
* to begin sketching the root loci of a system, we must know the location of the poles and zeros of $\underbrace{G(s)H(s)}_{\text{open loop.}}$

EX:

$$G(s)H(s) = \frac{k(s+Z_1)}{(s+P_1)(s+P_2)(s+P_3)(s+P_4)}$$



we draw a point s, and draw a line to each of the other points, and find the angle w/ respect to positive x axis.



$$|G(s)H(s)| = \frac{k|s+z_1|}{|s+P_1||s+P_2||s+P_3||s+P_4|}$$

$$= \frac{k\ B_1}{A_1 A_2 A_3 A_4}$$

if s is a closed loop pole (ie- s is a root of $1+GH=0$)

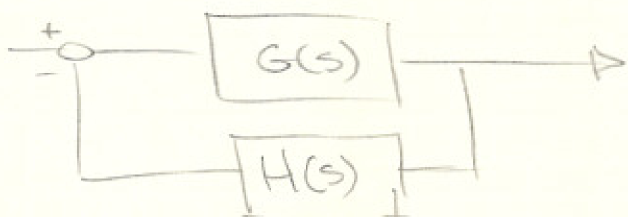
$$\angle G(s)H(s) = \angle s+z_1 + \angle s+P_1 + \angle s+P_2 + \angle s+P_2 + \angle S+P_4$$

$$= \pm 180^\circ(2k+1),\ k \in I$$

$$= \phi - \Theta_1 - \Theta_2 - \Theta_3 - \Theta_4$$

note: you do not have to plot both of the pathes of a set of conjugate poles because they are mirrored in the x axis.

RULES FOR ROOT LOCI PLOTTING.



$G(s)$ includes the plant + the controller

the characteristic equation.

$$1+G(s)H(s)=0 \quad (1)$$

Assume that $G(s)H(s) = \dfrac{k(s-a_1)(s-a_2)\ldots(s-a_n)}{(s-b_1)(s-b_2)\ldots(s-b_m)}$

$\therefore$ $a_1, a_2 \ldots a_n$ are the open loop zeros

& $b_1, b_2 \ldots b_n$ are the open loop poles

$① \Rightarrow (s-b_1)(s-b_2)\cdots(s-b_n) + k(s-a_1)(s-a_2)\cdots(s-a_n)$ ②

case 1:

for $k=0$ the closed loop poles coincide with the open loop poles.

set $k=0$ in ② $(s-b_1)(s-b_2)\cdot(s-b_n)=0$

case 2: for $k \to \infty$, the closed loop poles approach the open loop zeros.

set $k \to \infty$ in ② $(s-a_1)(s-a_2)\,(s-a_m)=0$

note: when $k=0$, the closed loop poles always start at the open loop poles, as k is increased to infinity, they approach the open loop zeros.

case 3: there are as many branches as there are open loop poles. A branch starts, for $k=0$, at each of the open loop poles. As k is increased, the closed loop poles trace out loci which end, for $k=\infty$, at the open loop zeros.

case 4: If there are fewer open-loop zeros than poles $(m<n)$, those branches for which there are no open loop zeros head to infinity along asymtopes. The number of asymtopes is equal to the number of open loop poles minus the number of open loop zeros $(n-m)$

case 5: the direction of the asymtopes are found from the angle condition. The asymtope angle α_k must satisfy

$$\alpha_k = \frac{\pm 180°(2k+1)}{n-m} \quad , k \in I$$

the angles are uniformy distributed over 360°

eg.

If $n-m=1 \Rightarrow \alpha$ is $180°$

$n-m=2 \Rightarrow \alpha$ is $90°, -90°$

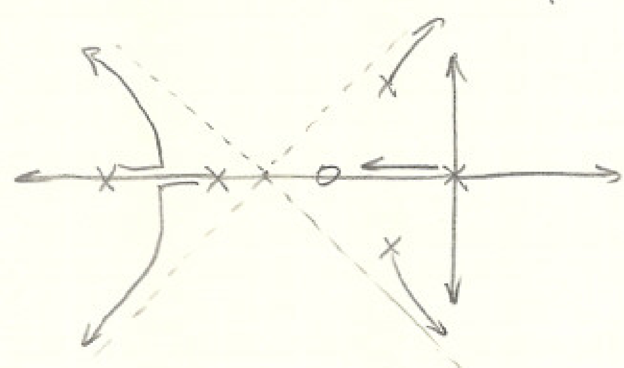
$n-m=3 \Rightarrow \alpha$ is $60°, -60°, 180°$

case 6: All assumptopes intersect the real axis at single point, at a distance of σ_0 to the origin

$$\sigma_0 = \frac{\sum \text{open-loop poles} - \sum \text{open-loop zerros}}{n-m}.$$

case 7: Loci are symetrical about the real axis since the complex open loop poles and zeros come in conjugate pairs.

case 8: Sections of the real axis to the left of an open loop poles & zeros on this axis from part of the loci



<u>case 9:</u> points of "breakaway from" or "arival at" the real axis between O.L. poles or (2 OL zeros) belongs to the loci, there must be a point between them where the loci break away from (arrive at) the real axis.

<u>EX:</u>

$$G(s)H(s) = \frac{k}{(s+2)(s+4)}$$

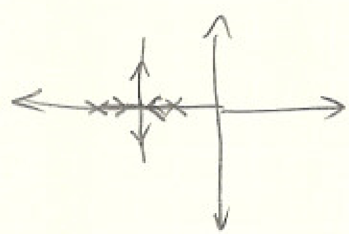
$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{(s+2)(s+4)} = 0$$

$$k = -(s+2)(s+4) = P(s)$$

$$\frac{dk}{ds} = \frac{dP(s)}{ds} = -(2s+6)$$

$$\therefore \quad s = -3 \quad \left(\text{Breakaway point}\right)$$



<u>note:</u> if there are no other poles or zeros near by then the breakaway point will be half way.

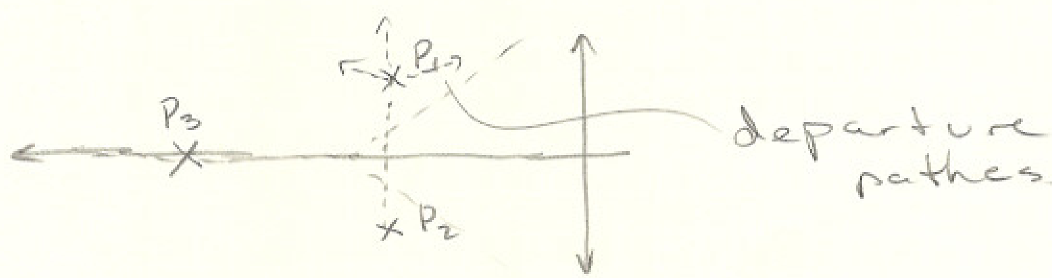
case 10: use the RH method to determine the point at which the locus crosses the imaginary axis (if it does so)

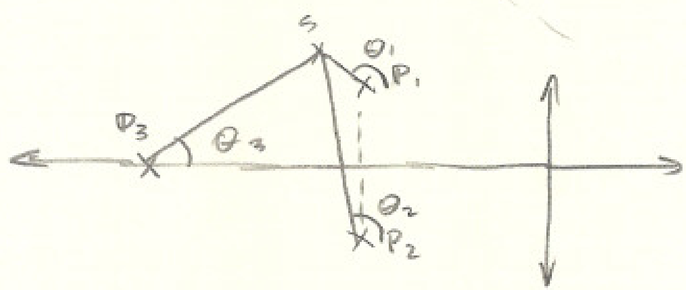
case 11: The angle of departure of the loci from the complex OL poles (or of arrival at complex OL zeros) is final significant feature. The angle of locus departure from a pole is the difference between the net angle due to all other poles and zeros. and the criterion angle of $\pm 180°(2k+1)$, and similairly for the locus angle of arival at zero.

EX:

$$G(s)H(s) = \frac{k}{(s+P_3)(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s-P_1)(s-P_2)$$





$$\theta_1 + \theta_2 + \theta_3 = 180° \quad \text{when } k=0$$

since s is very close to P_1, $\theta_2 \approx 90°$ and θ_3 can be obtain geometrically.