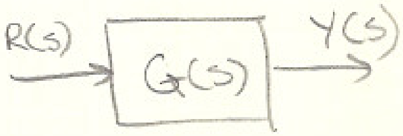
STEADY STATE ERROR

open loop system



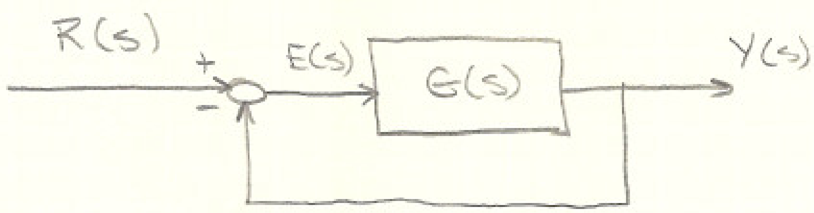
$$E(s) = R(s) - Y(s) = R(s) - R(s)G(s) = R(s)[1 - G(s)]$$

if $G(s)$ is stable.

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)[1 - G(s)]$$

if $R(s) = \frac{1}{s}$

$$\lim_{t\to\infty} e(t) = 1 - G(0)$$



$$G_c(s) = \frac{G(s)}{1 + G(s)}$$

$$E(s) = R(s) - Y(s) = R(s) - \frac{R(s)G(s)}{1 + G(s)}$$

$$= R(s)\left[\frac{1 + G(s) - G(s)}{1 + G(s)}\right] = \frac{R(s)}{1 + G(s)}$$

if $G(s)$ is stable

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)\frac{1}{1+G(s)}$$

if $R(s) = \frac{1}{s}$

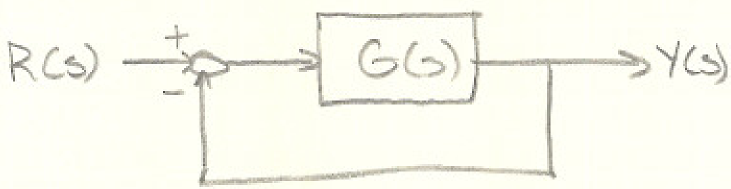
$$\lim_{t\to\infty} e(t) = \frac{1}{1+G(0)}$$

EX:

$$\frac{s+1000}{s^2+3s+1} \quad \text{(large DC gain)}$$

you can see that open loop error is large. However, closed loop will have a very small steady state error.

TYPE NUMBER OF SYSTEMS AND STEADY STATE.



$$sse = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

let
$$G(s) = \frac{K\prod_{i=1}^{m}(s+z_i)}{s^n \prod_{k=1}^{Q}(s+P_k)}$$

the number of integrations N, is called the TYPE NUMBER of the system.

<u>step input</u> for a type number of 0 (N=0) the sse for a step input of magnitude A, is:

$$sse = \lim_{s\to 0} \frac{sR(s)A}{1+G(s)} = \frac{A}{G(0)+1}$$

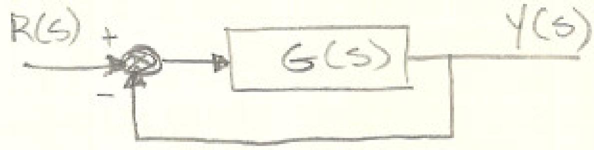
$$= \lim_{s\to 0} \frac{A}{1 + \dfrac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{k=1}^{n}(s+P_k)}} = 0$$

goes to ∞

G(0) (position error constant)

if $N \geq 1$, the steady state error is equal to zero.

RAMP INPUT.



$$G(s) = \frac{K \prod_{i=1}^{m} (s + z_i)}{s^n \prod_{k=1}^{Q} (s + P_k)}$$

$$R(s) = \frac{A}{s^2} \qquad r(t) = At$$

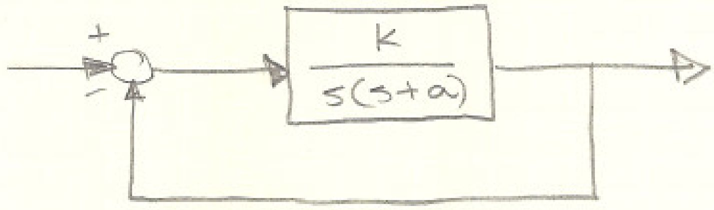
$$sse = \lim_{s \to 0} \frac{s(A/s^2)}{1 + G(s)} = \lim_{s \to 0} \frac{A}{s + sG(s)} = \lim_{s \to 0} \frac{A}{sG(s)}$$

* for a type zero $(N=0)$ the steady state error is infinity.
* for a type one $(N=1)$, the steady state error

$$\lim_{s \to 0} \frac{A}{sG(s)} = \frac{A}{\dfrac{K \prod_{i=1}^{m} z_i}{\prod_{k=1}^{Q} P_i}}$$

* if $N \geq 2$, then sse is zero.

EX:



characte eqn $= 1 + \frac{K}{s^2 + as} = 0$

$\therefore \quad s^2 + as + K = 0$

O.L. poles $= 0, -a$

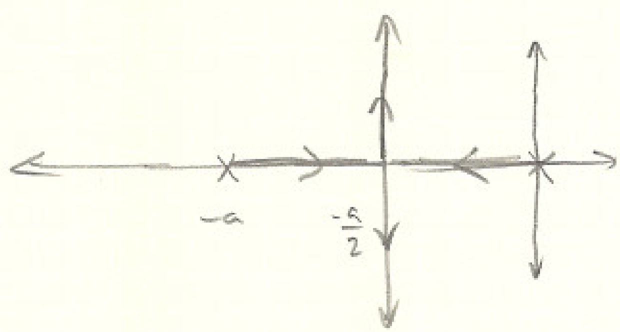
O.L. zeros = None

$n - m = 2 \Rightarrow 2$ asymtopes at 90° and -90°

Intersection of the asymtopes with

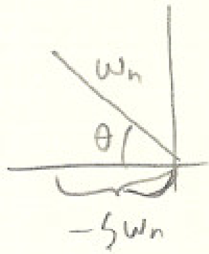
$$\sigma_0 = \frac{\Sigma \text{ OL POLES} - \Sigma \text{ OL ZEROS}}{n - m}$$

$$= -\frac{a}{2}$$



B.

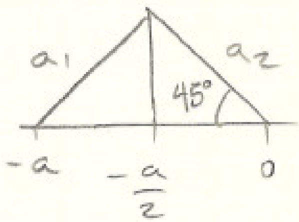
$\zeta = 0.7$



$$\cos\theta = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

$$\therefore \theta = 45°$$

now we can use geometry to obtain k



$$|G(s)H(s)| = 1 = \frac{k}{|s||s+a|}$$

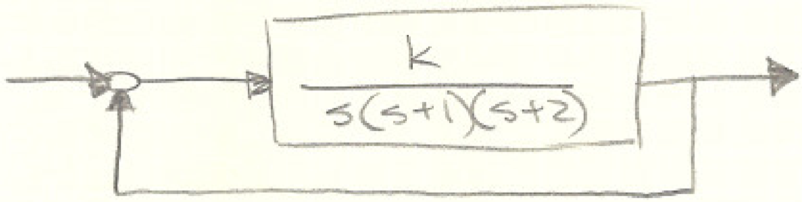
$$|s| = a_2 \quad |s+a| = a_1$$

$$a_1 \cdot a_2 = k$$

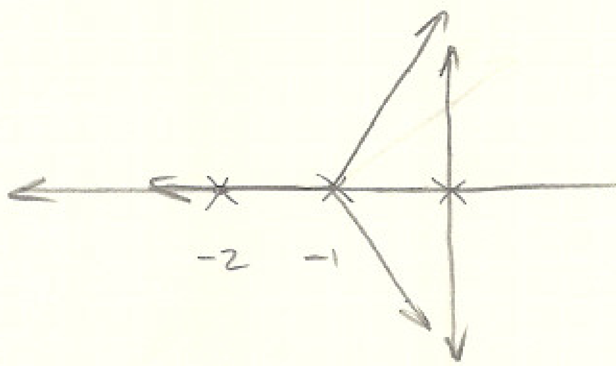
$$a_1 = a_2$$

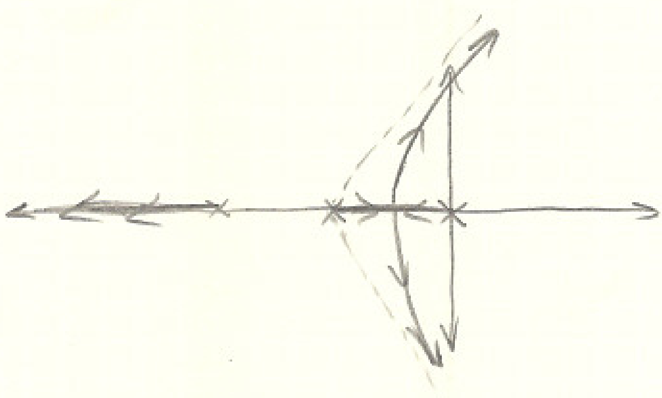
$$k = \frac{a^2}{2}$$

EX:



* OL POLES: 0, −1, −2
* OL ZEROS: NONE
* n − m = 3 ⇒ assumtopes, 60, 180, −60
* intersection of assumtopes = −1





we ott to also find the breakaway pt. and the intesection with the imaginary axis.

$$k = -s^3 - 3s^2 - 2s$$

$$\frac{dk}{ds} = -3s - 6s - 2$$

$\therefore$ $s = -1.57$ (excluded)

$s = -0.42$

intersection with the imaginary axis....

char eqn

$$0 = s^3 + 3s^2 + 2s + k$$

RH

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s & b_1 & \\ s^0 & c_1 & \end{array}$$

$b_1 = -\frac{1}{3}(k-6)$

$c_1 = k$

hence we know that when $k > 6$ system become unstable.

$$0 = s^3 + 3s^2 + 2s + 6$$

$$\therefore \quad s = \pm j\sqrt{2}$$

continuing from last example.

$$1+\frac{k}{s(s+1)(s+2)}=0$$

$$s^3+3s^2+2s+k=0$$

$$s=j\omega$$

$$\underbrace{-3\omega^2+K}_{=0}+\underbrace{j(2\omega-\omega^3)}_{=0}=0$$

$\therefore$

$$-3\omega^2+K=0 \qquad (1)$$

$$j(2\omega-\omega^3)=0 \qquad (2)$$

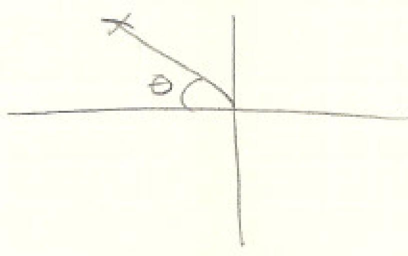
$$(2) \Rightarrow \omega(2-\omega^2)$$

$$\therefore \quad \omega=0$$

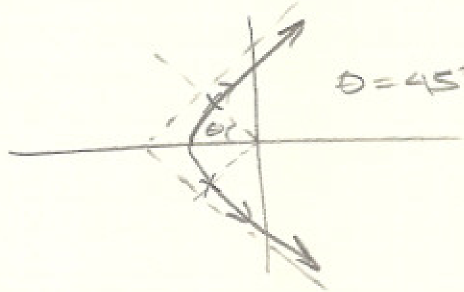
$$\omega=\pm\sqrt{2}$$

replace ω in (1) and find $k=6$.

Find "k" to realise a damping ratio 0.7, for the damping pair of poles of the CL system.



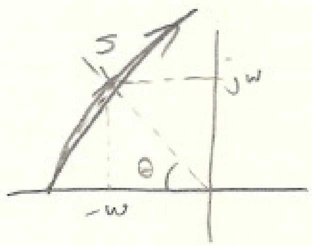
$$\zeta=\cos\theta$$



$\theta = 45^\circ \quad \therefore \ \zeta = 0.7$

any point on the line @ 45° to the negative real axis can be described by

$$s = \omega(-1 + j)$$



replace "s" by $s = \omega(-1+j)$

$$2(1+j)\omega^3 - 6j\omega^2 + 2\omega(-1+j) + K = 0$$

$$\underbrace{(2\omega^3 - 2\omega + K)}_{0 \qquad (2)} + j\underbrace{(2\omega^3 - 6\omega^2 + 2\omega)}_{0 \qquad (1)} = 0$$

$(1) \Rightarrow 2\omega(\omega^2 - 3\omega + 1) = 0$

$\therefore \ \omega = 0$

$\omega = 0.38$

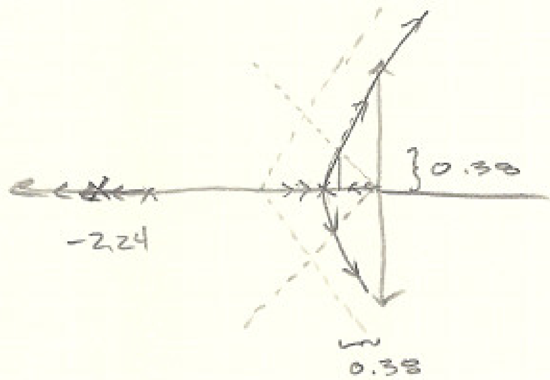
$\omega = 2.61$

for $\omega = 0.38 \Rightarrow (2) \Rightarrow K = 0.65$

for complex poles

$$P_{1,2} = -0.38 \pm j0.38$$

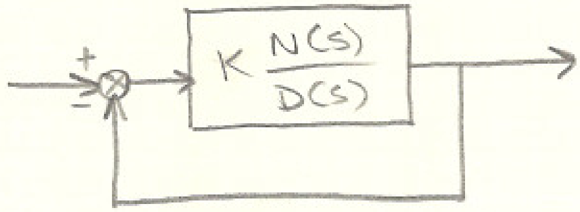
$$P_3 = -2.25$$



HW #4

B 6-3
B 6-6
B 6-10
B 7-18
B 7-19
B 7-20

ROOT LOCUS.



$N(s)=0$ results in O.L. zeros

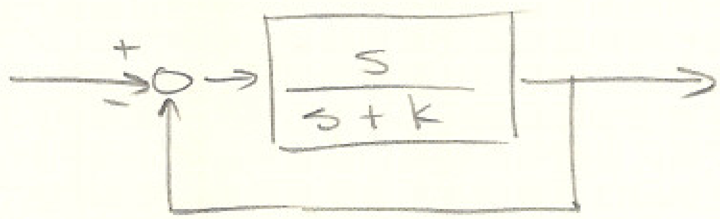
$D(s)=0$ results in O.L. poles.

$$1 + K\frac{N(s)}{D(s)} = 0$$

$$D(s) + KN(s) = 0$$

∴ closed loop poles are function of K.

Ex:

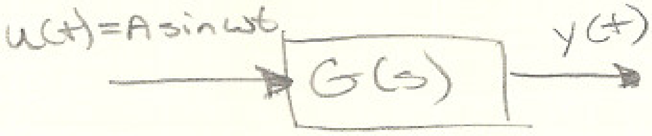


$$1 + \frac{s}{s+k} = 0 \Rightarrow \underbrace{2s}_{D(s)} + \underbrace{k}_{N(s)} = 0$$

∴ think.

FREQUENCY RESPONCE.

assume that system is stable.



what is $y(t)$ at steady state.

$$y(t) = B\sin(\omega t + \phi)$$

$$B = A\,|G(j\omega)|$$

$$\phi = \underline{\angle G(j\omega)}$$

Ex:

$$G(s) = \frac{1}{1+\tau s}$$

$$G(j\omega) = \frac{1}{1+\tau j\omega}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

$$B = \frac{A}{\sqrt{1+\tau^2\omega^2}}$$

$\omega \Rightarrow \infty \quad B \Rightarrow 0$

LP filter.

$$|G(j\omega)|_{dB} = 20 \log |G(j\omega)|$$

BODE PLOT.

the bode plot of a system $G(s)$ is:

- * The magnitude in dB verse $\log \omega$.
- * The phase in degrees verse $\log \omega$.

<u>note:</u>

$$\log(A \cdot B) = \log(A) + \log(B)$$

$$|G(j\omega)|_{dB} =$$

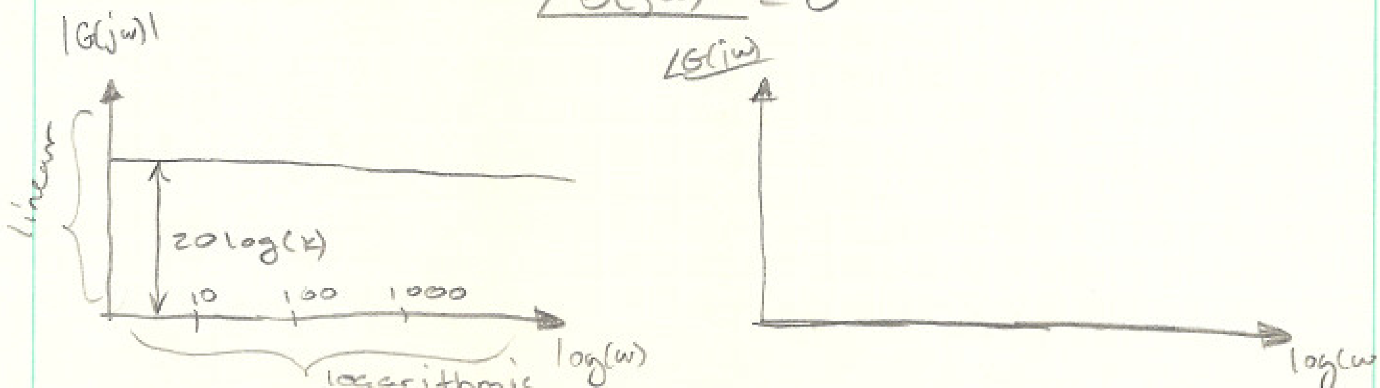
$$G(s) = \frac{1}{(s+1)(s^2+2s+3)}$$

$$= \left|\frac{1}{j\omega+1}\right|_{dB} + \left|\frac{1}{(j\omega)^2+2(j\omega)+3}\right|_{dB}$$

BODE PLOT FOR CONSTANT GAIN.

$G(s) = k \qquad |G(j\omega)| = 20\log(k)$

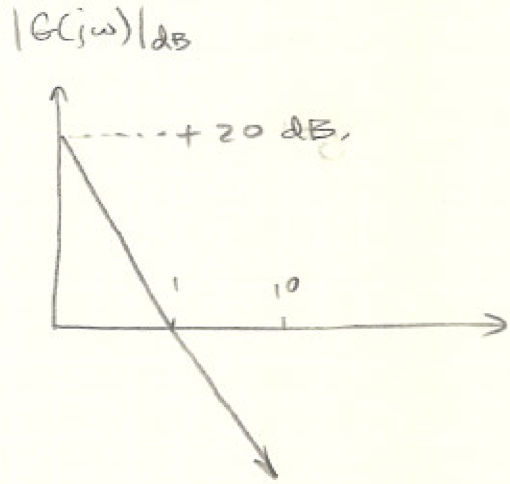
$\underline{\angle G(j\omega)} = 0$

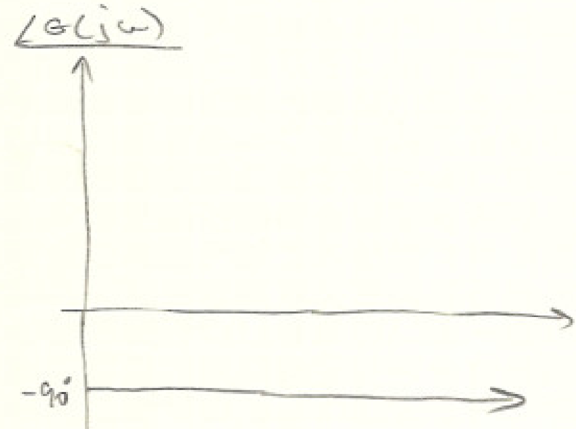


BODE PLOT FOR INTEGRATOR.

$$G(s) = \frac{1}{s} \qquad G(j\omega) = \frac{1}{j\omega}$$

$$|G(j\omega)| = \frac{1}{\omega} \qquad \underline{/G(j\omega)} = -90^\circ$$

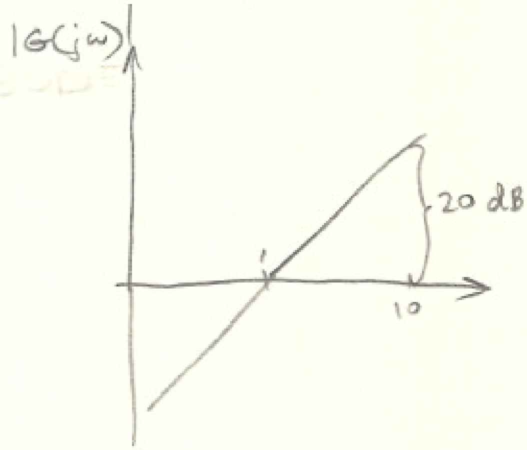


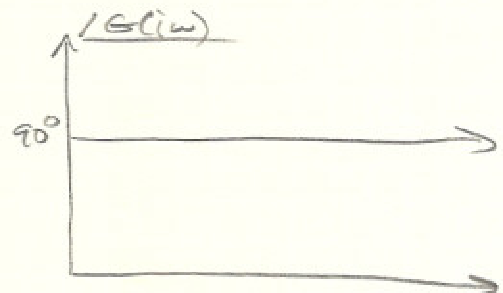


BODE PLOT FOR DERIVATIVE.

$$G(s) = s \qquad G(j\omega) = j\omega$$

$$|G(j\omega)| = \omega \qquad \underline{/G(j\omega)} = 90^\circ$$





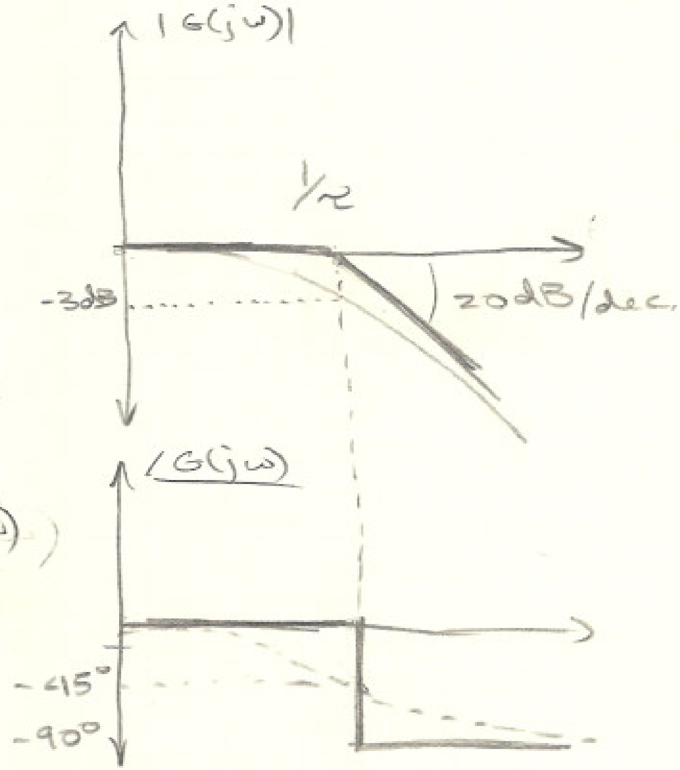
BODE PLOT OF LP FILTER

$$G(s) = \frac{1}{1+\tau s}$$

$$G(j\omega) = \frac{1}{1+\tau j\omega}$$

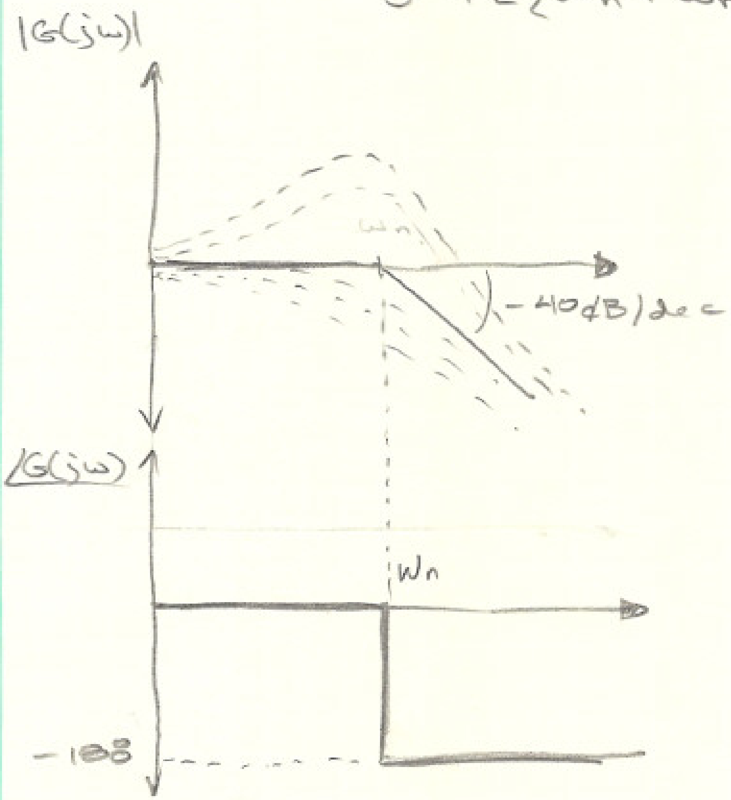
$$|G(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

$$\angle G(j\omega) = -\tan^{-1}(\tau\omega)$$

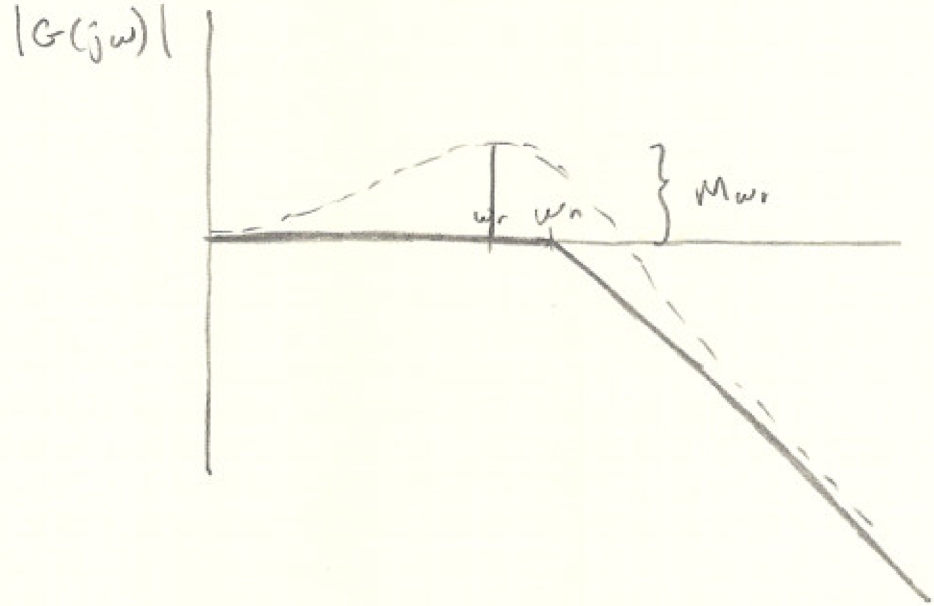


BODE OF SECOND ORDER

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$



note: differences come from damp ratio



ω_r: resonance freq

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

$$M_{\omega r} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

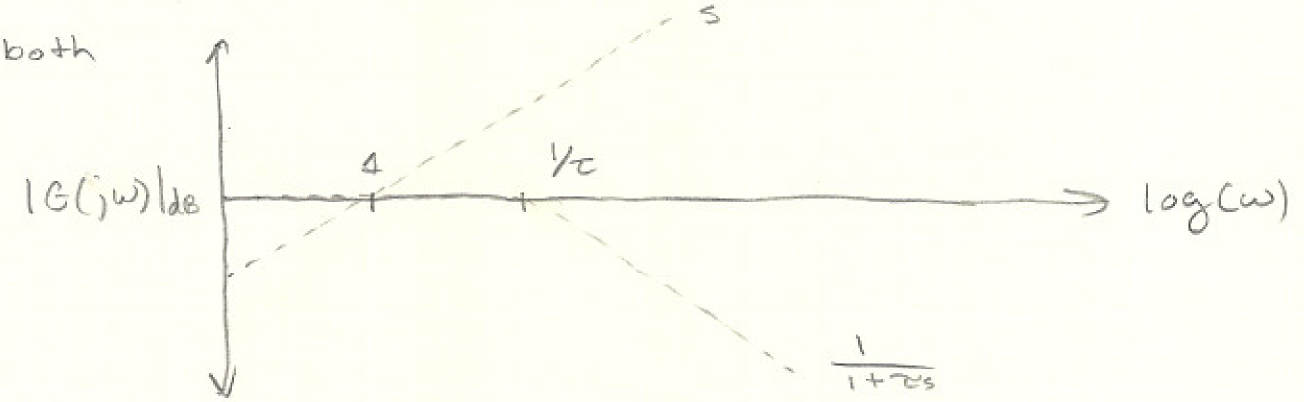
for $0 < \zeta < 0.707$

if $\zeta > 0.707$, there is no peak

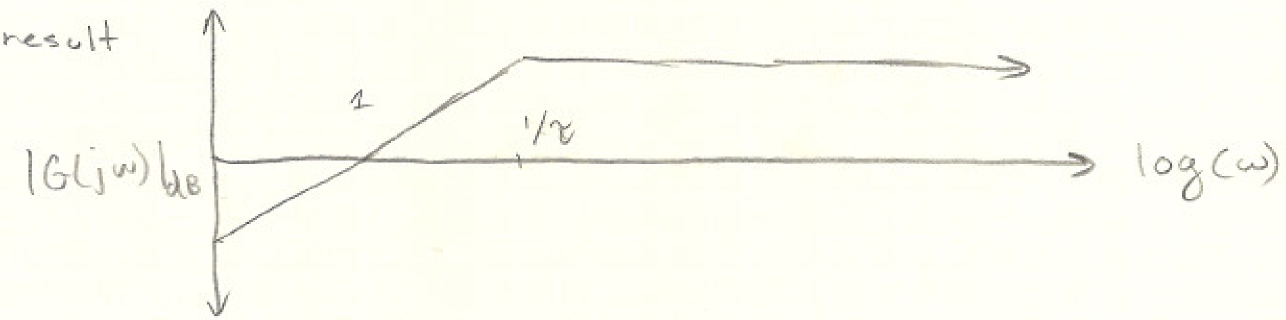
BODE DIAGRAM FOR $\frac{s}{1+\tau s}$

<u>note</u>: for bode, we can add results.

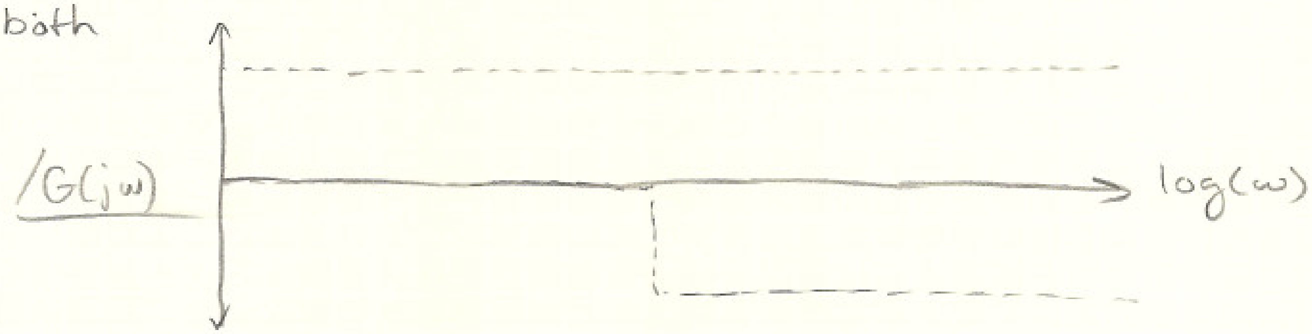
both



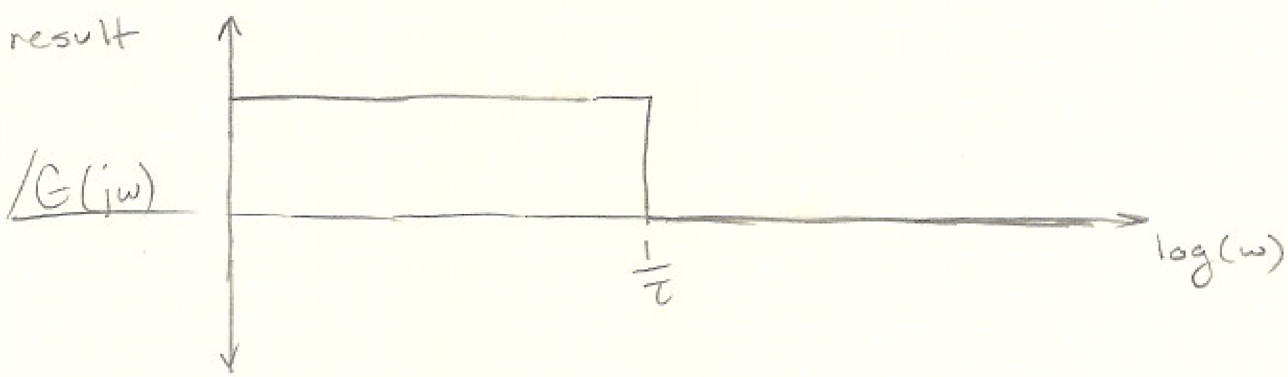
result



both



result

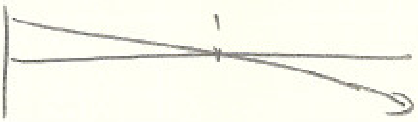


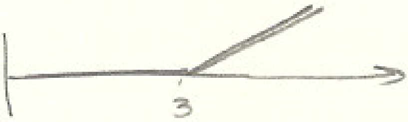
Ex:

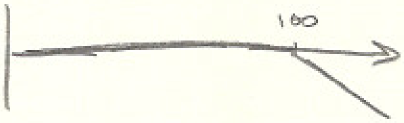
$$G(s) = \frac{30\left(\frac{1}{3}s+1\right)}{100s\left(\frac{s}{100}+1\right)}$$

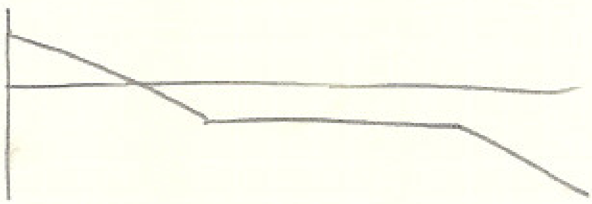
$$G(s) = \underbrace{\frac{3}{10}}_{①} \cdot \underbrace{\left(\underbrace{\frac{1}{3}}_{\tau_1}s+1\right)}_{②} \cdot \underbrace{\frac{1}{s}}_{③} \cdot \underbrace{\frac{1}{\left(\underbrace{\frac{s}{100}}_{\tau_2}+1\right)}}_{④}$$

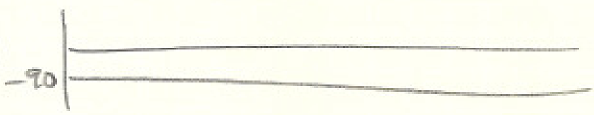
① : $20\log\left(\frac{3}{10}\right) = -0.5$

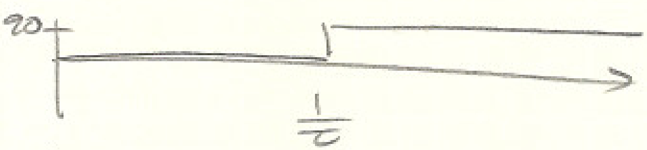
③


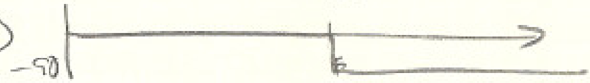
②


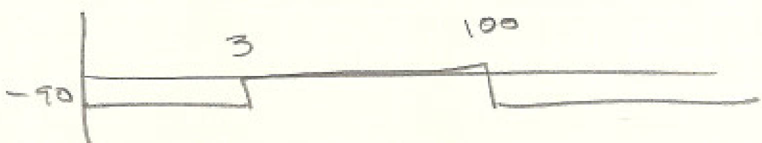
④


result


③


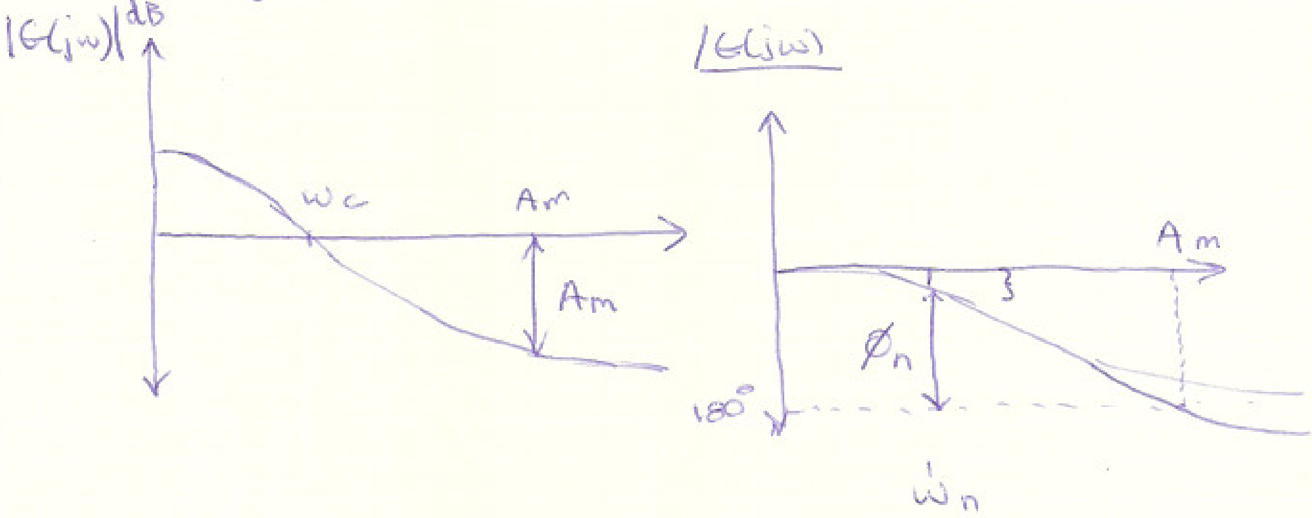
②


④


result


IDENTIFICATION.

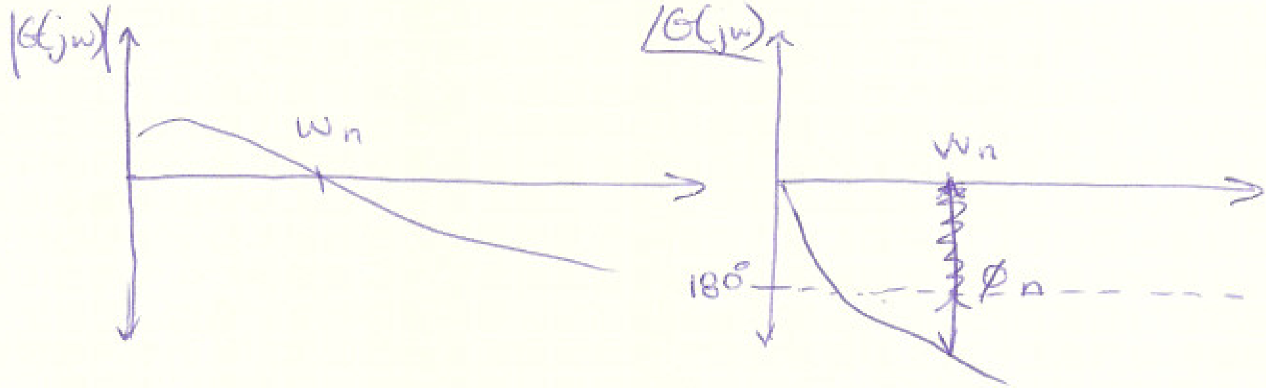
finding G(s) from BODE PLOT.



A_m: gain margin

W_n: ϕ_n.

note: this system is ~~un~~stable.



$\phi_n < 0$ ∴ unstable.

Conditions for stability

$\phi_n < 0$ unstable

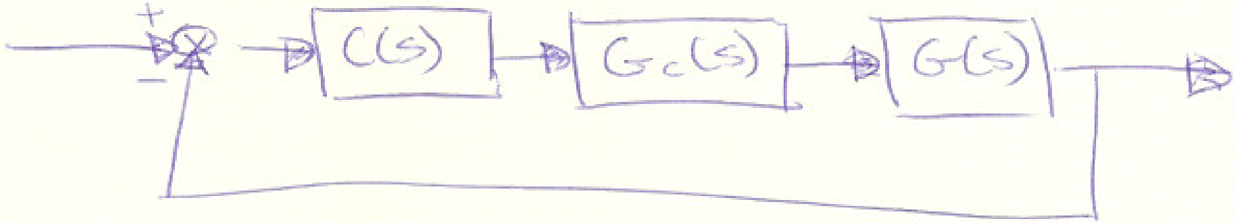
$\phi_n > 0$ stable.

for good design. $\phi_m \approx 45°$

note: controllers can be used to increase either chart, gain or phase.

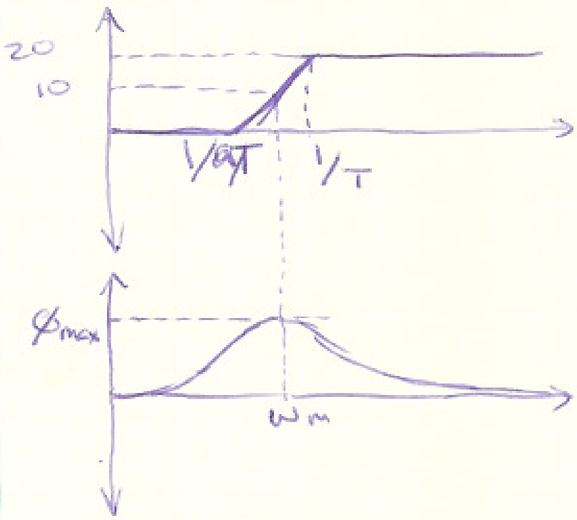
PHASE LEAD COMPENSATION.

assume that $G(s)$ is stable



assume $G(s)$ is stable

$$G_c = \frac{1 + aTs}{1 + Ts} \qquad a > 1$$



$$\omega_{max} = \frac{1}{\sqrt{a}\,T}$$

$$\sin \phi_{max} = \frac{a-1}{a+1}$$

$$a = \frac{1 + \sin \phi_{max}}{1 - \sin \phi_{max}}$$

PROCEEDURE

1. THE BODE DIAGRAM OF THE COMPENSATED PROCESS G(S) IS CONTRACTED, WITH THE GAIN K SET ACCORDING TO THE SSE REQUIREMENTS.

$$sse = \left| 1 - \lim_{s \to 0} sY(s) \right| = \left| 1 - \frac{kG_c(0)G(0)}{1 + kG_c(0)G(0)} \right|$$

$$= \left| \frac{1}{1 + kG_c(0)} \right|$$

DESIGN K

Plot the bode diagram of $KG(j\omega)$

2. The phase margin and the gain margin of the uncompensatied system are read from the bode plot, and the additional amount of phase lead needed to realise the designed phase margin is determined. as ϕ_{max}.

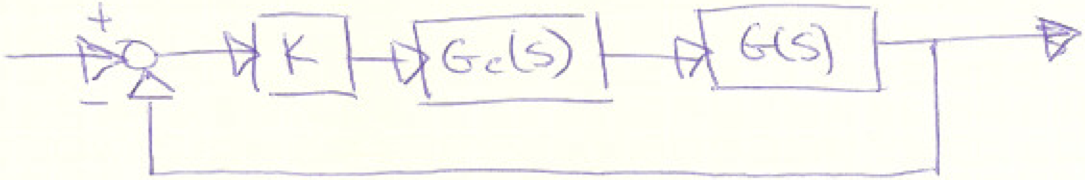
3. EVALUATE

$$a = \frac{1 + \sin \phi_{max}}{1 - \sin \phi_{max}}.$$

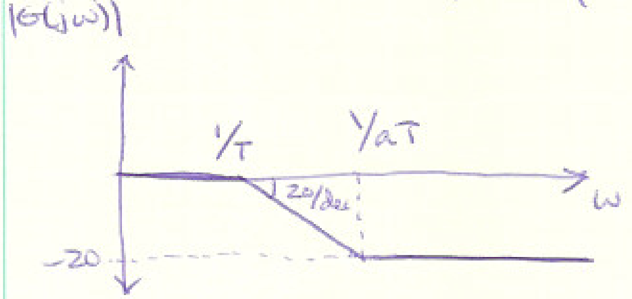
4. Evaluated $10 \log a$. and determine the frequency where the uncompensated magnitude curve is equal to $-10 \log a$ (dB). B/c the compensater ~~again~~ provides gain of $10 \log a$ at ω_{max}, ~~this~~ frequency is the new 0 dB crossover freq.

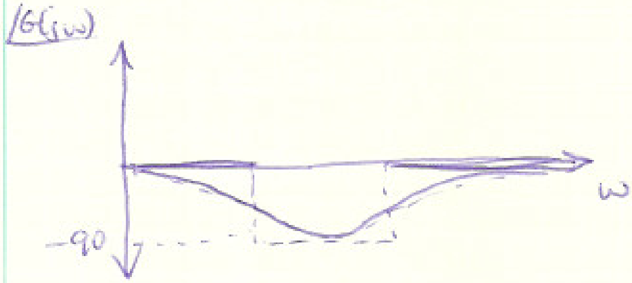
5. Evalvate $\tau = \frac{1}{\omega_{max}\sqrt{a}}$

Phase lag Compensator



$$G_c(s) = \frac{1 + aTs}{1 + Ts} \qquad a < 1$$





$$\sin\phi_{max} = \frac{a-1}{a+1} \quad \therefore \quad a = \frac{1+\sin\phi_{max}}{1-\sin\phi_{max}}$$

PROCEEDURE

1. Design K for steadystate error requirements
2. plot the bode diagram of KG(s)
3. Determine the phase margin for the uncompensated system (ie. KG(s)) and if it is insufficent proceed withe the following
4. Determine the freq where the phase margin required would be satified if the

magnitude curve crossed the 0 dB line at this freq ω'_c

5. to bring the magnitude curve down to 0dB. at the new crossover freq ω'_c, the phase lag controller must provide the ammount of attenation equal to the value of the magnitude curve ω'_c.

$$a = 10^{-|KG(j\omega)|/20} \qquad a < 1$$

6. $T = \dfrac{10}{a\omega'_c}$

ACCELERATION INPUT.

$r(t) = A t^2$

$R(s) = \frac{A}{s^3}$

and from here it is clear that when

$N = 0, 1$ $\quad$ $sse = \infty$
$N = 2$ $\quad$ $sse = \text{Constant}$
$N \geq 3$ $\quad$ $sse = 0$

STABILITY OF LINEAR TIME INVARIANT SYSTEMS.

a dynamic system is stable if a bounded input produces a bounded output.

A dynamic system is unstable if a bounded input generates a unbounded output.

* a system is stable iff all the poles in the transfer function describing the system are on the right hand side of the imaginary axis.

* a system is unstable when there is a pole in the right hand plane.

* A system is marginally stable if it's transfer function has a simple pole on the imaginary axis and all the other poles are on the left hand plane

For marginally stable systems, the output is unbounded for bounded inputs having the same freq as the pole on the imaginary axis. (the output is bounded for inputs of other freq's)

if $\omega = \omega$, then the system becomes unstable.

ROUTH - HURWITZ STABILITY CRITERON

let us consider the characteristic equation of an n^{th} order LTI system:

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0 \qquad (1)$$

if all the parmaters $\{a_i\}$ are negative, it is possible to eliminate the $\{-\}$ sign from the characteristic equ (1) to obtain all the parameters with a $\{+\}$ sign.

* all terms in equ (1) must be positive. (to keep all the poles in the left hand plane).

construct an array.

$$\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_{n-2} & b_{n-3} & b_{n-5} & \cdots \\
\vdots & & \vdots & & \\
s^0 & & & &
\end{array}$$

$$b_{n-1} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-3} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

The RH criterion states that the number of roots of the characteristic equation (1) with the positive real part equal equal to the number of changes in sign of the first column of the right hand column.

CASE 1: (no element in the first column of the right hand array is zero.)

$$a_2 s^2 + a_1 s + a_0 = 0$$

$$\begin{array}{c|cccc} s^2 & a_2 & a_0 & \emptyset & \emptyset \\ s^1 & a_1 & \emptyset & \emptyset & \emptyset \\ s^0 & b_1 & & & \end{array}$$

$$b_1 = \frac{-1}{a_1} \begin{vmatrix} a_2 & a_0 \\ a_1 & \emptyset \end{vmatrix} = a_0$$

a characteristic eqn is stable iff

$$a_2 > 0$$

$$a_1 > 0$$

$$a_0 > 0$$

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ \hline s^1 & b_1 & b_2 \\ s^0 & c_1 & c_2 \end{array}$$

b_2 & c_2 are zero.

$$b_1 = \frac{-1}{a_2} \begin{vmatrix} a_3 & a_1 \\ a_2 & a_0 \end{vmatrix} = \frac{-1}{a_2}(a_3 a_0 - a_1 a_0)$$

$$c_1 = \frac{-1}{b_1} \begin{vmatrix} a_2 & a_0 \\ b_1 & b_2 \end{vmatrix} = a_0$$

therefore the system is stable iff.

$$\left.\begin{matrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{matrix}\right\} > 0$$

$$a_1 a_2 - a_3 a_0 > 0$$