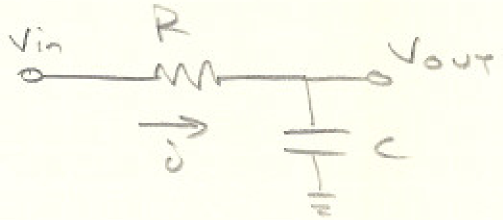
PERFORMANCE OF A FIRST ORDER SYSTEM.

$$\frac{Y(s)}{R(s)} = \frac{1}{1+\tau s}$$

$\tau > 0$ (time constant.)

EX:



$$V_{IN} = Ri + V_{OUT} \qquad (1)$$

$$V_{OUT} = \frac{1}{C}\int i\,dt \qquad (2)$$

$$(2) \Rightarrow \quad i = C\frac{dV_{OUT}}{dt} \qquad (3)$$

$$(2) \rightarrow (1) \quad V_{in} = RC\frac{dV_{OUT}}{dt} + V_{OUT} \qquad (4)$$

$$\tau = RC \qquad (5)$$

$$(5) \rightarrow (4) \rightarrow \mathcal{L} \quad V_{in} = \tau s\, V_{OUT}(s) + V_{OUT} \qquad (6)$$

$$(6) \Rightarrow \quad \frac{V_{OUT}}{V_{in}} = \frac{1}{1+\tau s.} \qquad (7)$$

$$\frac{Y(s)}{R(s)} = \frac{1}{s+a} \qquad (1)$$

$$(1) \Rightarrow \frac{Y(s)}{R(s)} = \frac{1}{a} \cdot \frac{1}{\frac{s}{a}+1} \qquad (2)$$

$$\tau = \frac{1}{a} \qquad (3)$$

$$(3) \rightarrow (2) \quad \frac{Y(s)}{R(s)} = \frac{\frac{1}{a}}{\tau s + 1} \qquad (4)$$

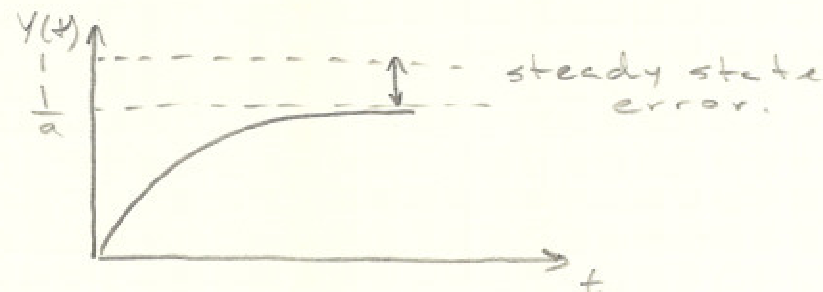
Ex:

$$r(t) = \mathcal{L}^{-1}\left[R(s)\right] = 1 \qquad (1)$$

$$(1) \Rightarrow \quad R(s) = \frac{1}{s} \qquad (2)$$

$$(2) \rightarrow (k) \quad Y(s) = \frac{1}{s+a} \cdot \frac{1}{s} \qquad (3)$$

$$(3) \rightarrow \mathcal{L}^{-1} \quad y(t) = \frac{1}{a}\left(1 - e^{-at}\right) \qquad (4)$$

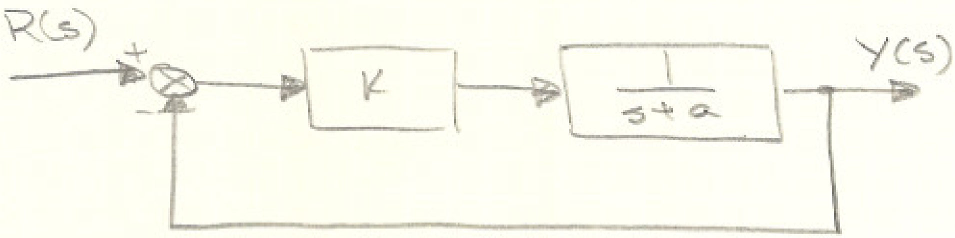


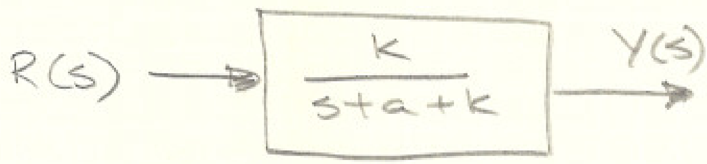
we can use the final value theorem to predict the steady state error

$$\lim_{t \to 0} y(t) = \lim_{s \to 0} sY(s) = s \cdot \frac{1}{s} \cdot \frac{1}{s+a} = \frac{1}{a}$$

STEADY STATE ERROR $= \left| 1 - \frac{1}{a} \right|$

EX: this time we will be adding gain.





assume that $a<0 \Rightarrow$ open loop system is unstable.

$k+a>0 \Rightarrow$ system becomes stable.

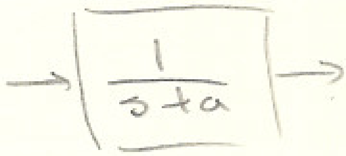
closed loop steady state.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{k}{k+a}$$

$$sse = \left| 1 - \frac{k}{k+a} \right|$$

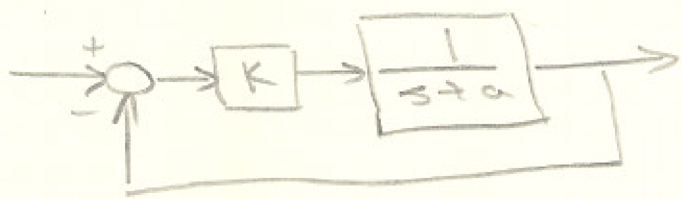
you can observe that by i/c k we reduce sse.

from last day



$SSE = \left| 1 - \frac{1}{a} \right|$

but if we apply amplified feedback.



then $SSE = \left| 1 - \frac{k}{k+a} \right|$

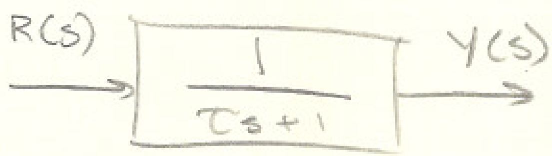
here $G_c(s) = \dfrac{k}{s+a+k} = \dfrac{k}{a+k} \cdot \dfrac{1}{\tau s+1}$

where $\tau = \dfrac{1}{a+k}$

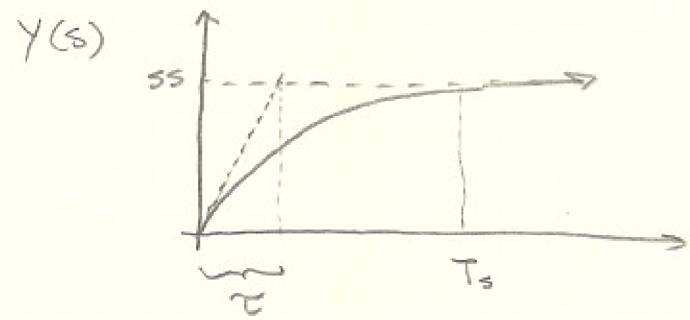
note: the smaller τ becomes the faster the system becomes.

PERFORMANCE OF A FIRST ORDER SYSTEM.

$G(s) = \dfrac{1}{\tau s + 1}$ $\tau > 0$ (time constant of the system)



where $R(s) = \frac{1}{s}$



$T_s \cong 5\tau$

where T_s is the settling time.

PERFORMANCE OF A 2nd ORDER SYSTEM.

a stable second order system is given by

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where $\omega_n > 0 \rightarrow$ natural freq
$\xi > 0 \rightarrow$ damping freq

note: by increasing ξ, then we are damping out ocsillations.

we have 3 cases depending on the value of ξ

1 → $\xi > 1$ (overdamped)

2 → $\xi = 1$ (critically damped system)

3 → $0 < \xi < 1$ (underdamped system)

CASE 1: the roots of this equ are real and distinct:

these roots are $P_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

$$\therefore \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{(s - P_1)(s - P_2)}$$

when $R(s) = 1/s$

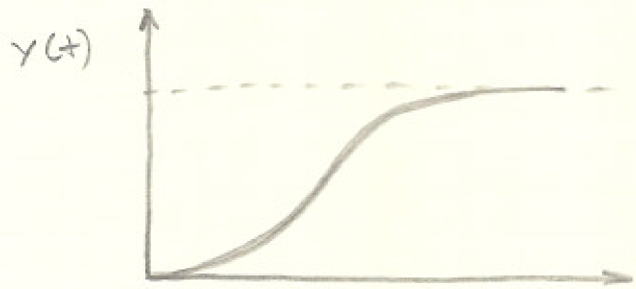
$$Y(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{(s-P_1)(s-P_2)} = \frac{k_1}{s} + \frac{k_2}{s-P_1} + \frac{k_3}{s-P_2}$$

where $k_1 = \dfrac{\omega_n^2}{P_1 P_2}$ $\quad k_2 = \dfrac{\omega_n^2}{P_1(P_1-P_2)}$ $\quad k_3 = \dfrac{\omega^2}{P_2(P_2-P_1)}$

$$\therefore \quad y(t) = k_1 + k_2 e^{P_1 t} + k_3 e^{P_2 t}$$

$\underline{\underline{P_1 < 0 \; , P_2 < 0}}$ $\quad$ & $\quad P_1 \times P_2 = 1 \Rightarrow k_1 = 1$

then the steady state is 1



CASE 2: the roots are Real but the same.

$$P_1 = P_2 = -\zeta \omega_n$$

$$\therefore \quad \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{(s-P_1)^2} \Rightarrow Y(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{(s-P_1)^2}$$

$$y(t) = \frac{\omega_n^2}{P_1^2}\left[1 - e^{P_1 t}(1 - P_1 t)\right]$$

$$\frac{\omega_n^2}{P_1^2} = 1 \quad \therefore \quad y(t) = 1 - e^{P_1 t}(1 - P_1 t)$$

CASE 3: where the roots are complex.

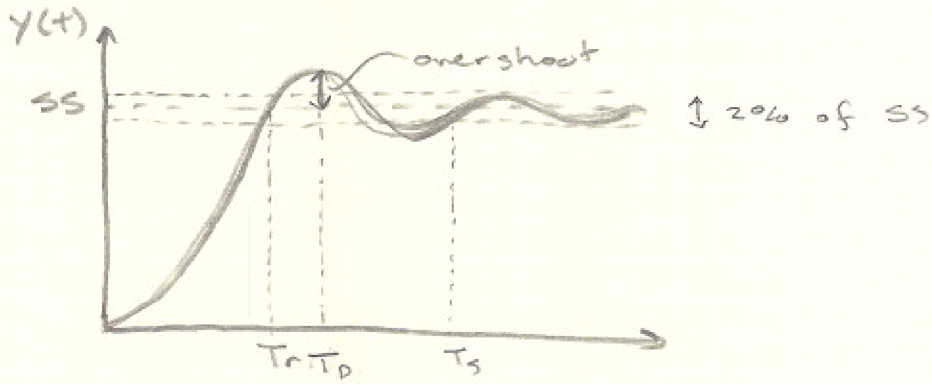
$$P_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

the unit step responce is given by

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\beta t} \sin(\omega_n\beta t + \phi)$$

where $\beta = \sqrt{1-\zeta^2}$

$\phi = \cos^{-1}(\zeta)$



note: we say the system is settled when it is in within 2% of the final value.

note:

$$T_s = \frac{4}{\zeta\omega_n}$$

note: % overshoot $= 100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$

* know by heart *

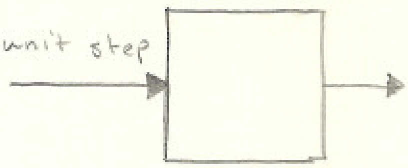
note: $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

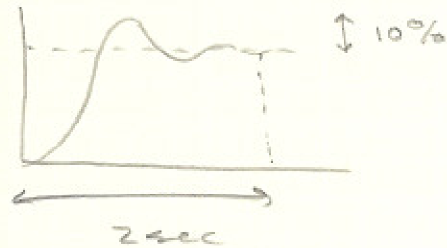
note: Peak responce
$MP_1 = 1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$

note: $T_r = \dfrac{\pi - \cos(\zeta)}{\omega_n\sqrt{1-\zeta^2}}$

EX: Determine the TF of a secound order system with 10% overshoot and a settling time with 2% of 2 seconds.

$$G(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$





$$P.O. = 100\, e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 10$$

$$\frac{-\zeta\pi}{\sqrt{1-\zeta^2}} = \ln(0.1)$$

$$\frac{\zeta^2\pi^2}{1-\zeta^2} = (\ln(0.1))^2$$

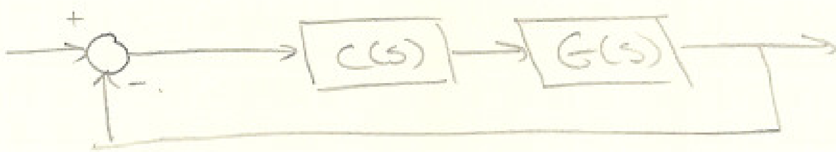
$$\therefore \quad \zeta = \sqrt{\frac{\ln(0.1)^2}{\pi^2 + (\ln(0.1))^2}} = 0.591$$

settling time

$$T_s = \frac{4}{\zeta\omega} = 2 \qquad \therefore \quad \omega = 3.3832$$

and

$$G(s) = \frac{11.4461}{s^2 + 4s + 11.4461}$$



$$G_c(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

and we wand to make

$$G_c(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

this is hard to do.

$$C(s) = \frac{N_c}{D_c} \qquad G(s) = \frac{N_G}{D_G}$$

$\therefore$ matching the denominator.

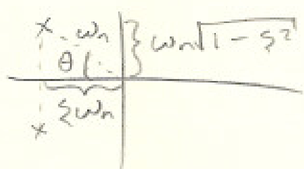
$$1 + G(s)C(s) = 1 + \frac{N_c}{D_c} \cdot \frac{N_G}{D_G} = 0$$

$$D_c D_G + N_c N_G = 0 \qquad (1)$$

the roots of (1) are the poles of the closed loop system

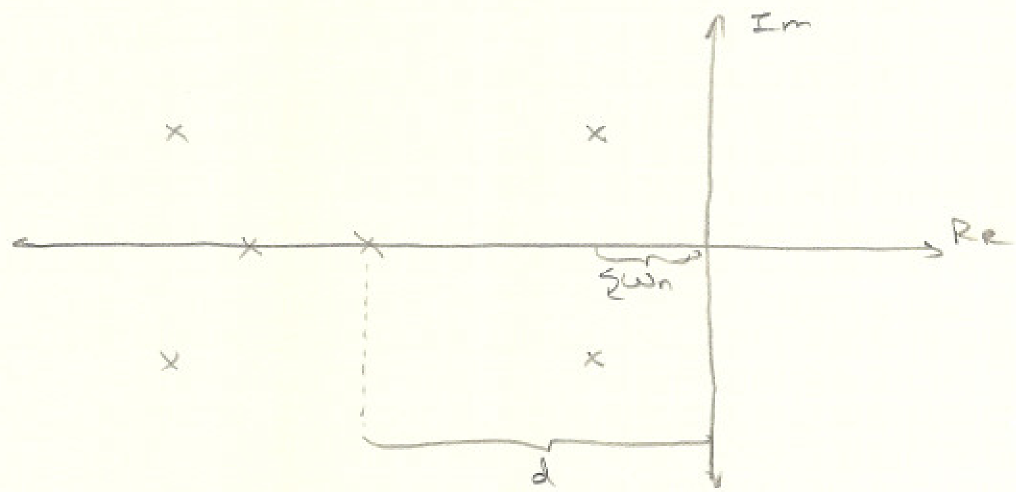
now find N_c & D_c such that

$$D_c D_G + N_c N_G = s^2 + 2\zeta\omega s + \omega^2$$

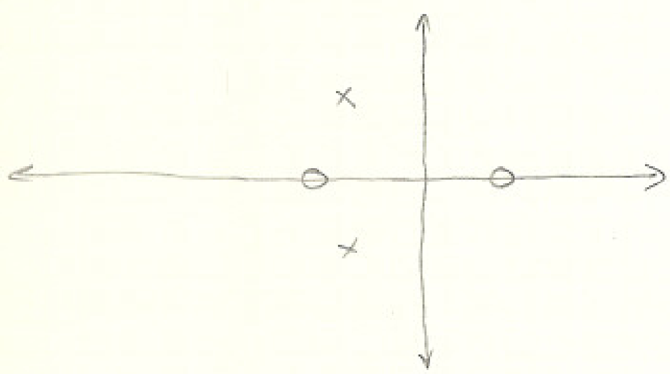


EFFECTS OF ZEROS & ADDITIONAL POLES.

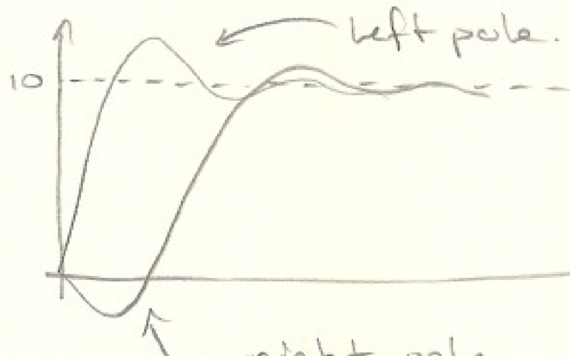
<u>note</u>: the poles that are closest to the imaginary axis are the most dominate poles.



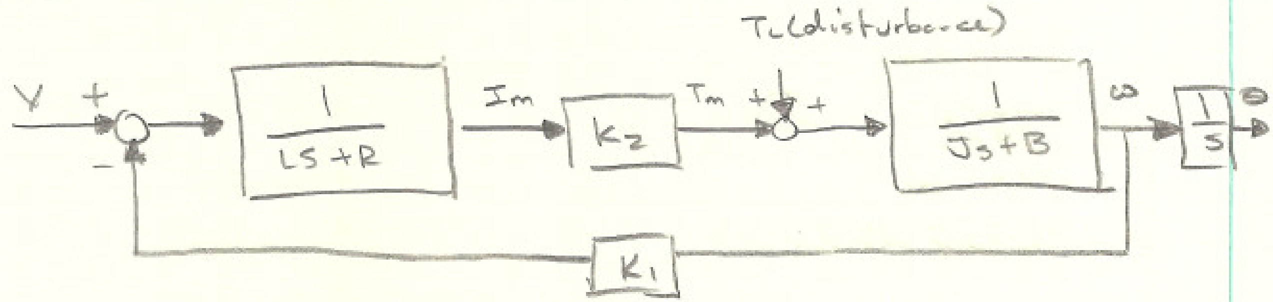
$d > 5\zeta\omega_n$, then we can say the system can be aprovimated by closer poles.



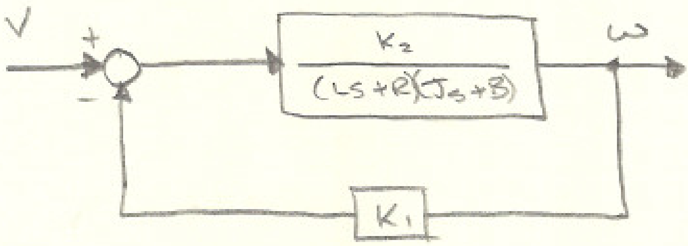
the zero's generally effect the undershoot of the system.



<u>note</u>: this is the characteristic of a pole in right hand plane.

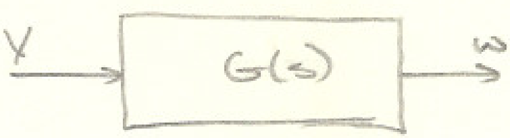


Assume that $T_L = 0$



$\frac{L}{R}$: electrical time constant

$\frac{J}{B}$: mechanical time constant.



$$G(s) = \frac{\frac{k_2}{(Ls+R)(Js+B)}}{1+\frac{k_1k_2}{(Ls+R)(Js+B)}}$$

$$= \frac{k_2}{RB(\frac{J}{B}s+1)+k_1k_2}$$

$$= \frac{k_2}{RJS+RB+k_1k_2}$$

pole is

$$s = \frac{-RB - k_1 k_2}{RJ}$$

which will be less then 0, b/c all the values are positive.

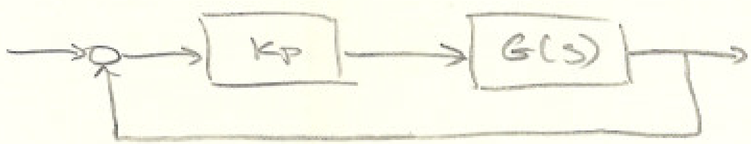
assume that the input is a unit step.

$$\lim_{t \to \infty} \omega(t) = \lim_{s \to 0} s\,W(s) = \lim_{s \to 0} s\,G(s)\frac{1}{s} = G(0)$$

$$sse = \left| 1 - \frac{k_2}{RB + k_1 k_2} \right|$$

to eliminate sse, we add a porportional controller

$$G(s) = \frac{k_2}{RJs + RB + k_1 k_2}$$



$$G_c(s) = \frac{\dfrac{k_p k_2}{RJs + RB + k_1 k_2}}{1 + \dfrac{k_p k_2}{RJs + RB + k_1 k_2}}$$

$$= \frac{k_p k_2}{RJs + RB + k_1 k_2 + k_p k_2}$$

$$G_c(s) = \frac{(k_p k_2)(RB + k_1 k_2 + k_p k_2)}{\tau s + 1}$$

where $\tau = \dfrac{RJ}{RB + k_1 k_2 + k_p k_2}$

by increasing k_p, we decrease τ, making the system fast.

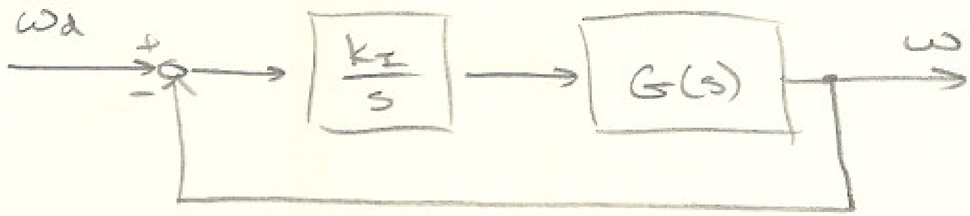
for the closed loop system, the sse is

$$\left| 1 - \frac{k_p k_2}{RB + k_1 k_2 + k_p k_2} \right|$$

by increasing k_p, we decrease the steady state error.

$$sse = 0 \text{ for } k_p \to \infty$$

CONSIDER INTEGRAL CONTROLLER



$$G_c(s) = \frac{\dfrac{k_I k_2}{s(RJs + RB + k_1 k_2)}}{1 + \dfrac{k_I k_2}{s(RJs + RB + k_1 k_2)}}$$

$$= \frac{k_I k_2}{s(RJs + RB + k_1 k_2) + k_I k_2}$$

we can see here that the steady state error is zero, b/c of the integral action.

$$= \frac{k_I k_2}{RJs^2 + (RB + k_1 k_2)s + k_I k_2}$$

put this in the form of a standard first order system

$$G_c(s) = \frac{(k_I k_2)/RJ}{s^2 + \left(\frac{RB + k_1 k_2}{RJ}\right)s + \left(\frac{k_I k_2}{RJ}\right)}$$

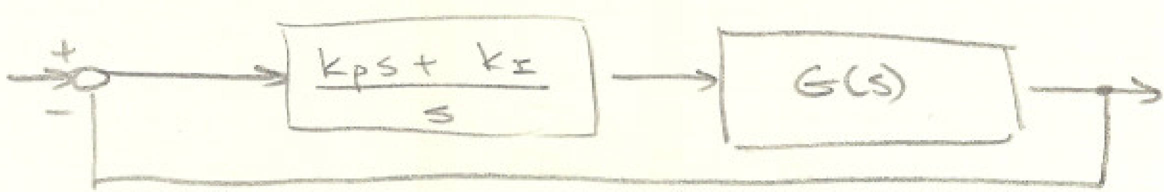
$$2\zeta\omega_n = \frac{RB + k_1 k_2}{RJ}$$

$$\omega_n^2 = \frac{k_I k_2}{RJ}.$$

note: with intergral control, we can not control settling time.

CONSIDER PI CONTROLLER

same system, but this time it will be controlled with a PI controller



$$G_c(s) = \frac{\left(\frac{k_p s + k_I}{s}\right)\left(\frac{k_2}{RJs + RB + k_1 k_2}\right)}{1 + \left(\frac{k_p s + k_I}{s}\right)\left(\frac{k_2}{RJ + RB + k_1 k_2}\right)}$$

$$C(s) = \frac{k_p s + k_i}{s} = \frac{k_p\left[s + \frac{k_i}{k_p}\right]}{s}$$

$$G(s) = \frac{k_2/RJ}{s + \frac{RB + k_1k_2}{RJ}}$$

let $\quad \frac{k_i}{k_p} = \frac{RB + k_ik_2}{RJ} \qquad (1)$

now closed loop looks like.

$$G_c(s) = \frac{\frac{k_pk_2}{RJs}}{1 + \frac{k_pk_2}{RJs}} = \frac{k_pk_2}{RJs + k_pk_2}$$

results in Ø sse.

$$G_c(s) = \frac{1}{\tau s + 1}, \quad \text{where } \tau = \frac{RJ}{k_pk_2}$$

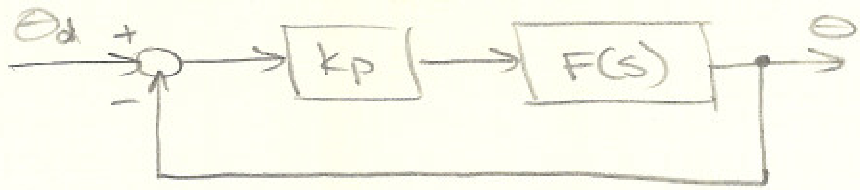
$$T_s = 5\tau$$

$$T_s = \frac{5RJ}{k_pk_2} \qquad (2)$$

note: combined the proportional and integral controls, we can control the rise time and steady state error.

CONSIDER CONTROLLING THE POSITION OF THE SYSTEM.

PROPORTIONAL CONTROLLER.



$$F_c(s) = \frac{k_p k_2}{RJs^2 + (RB + k_1 k_2)s + k_p k_2}$$

we can see that the sse is ∅

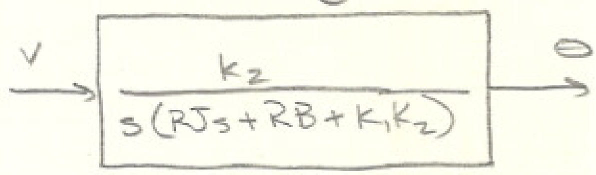
$$F_c(s) = \frac{k_p k_2 / RJ}{s^2 + \left(\frac{RB + k_1 k_2}{RJ}\right)s + \frac{k_p k_2}{RJ}}$$

but we can see we have no control over the settling time. we will need to use derivative control.

PROPORTIONAL DERIVATIVE CONTROLLER.

$F_c(s) =$

from last day.



$G(s)$

PD control system

$$C(s) = k_P + k_D s$$

$$C(s)\,G(s) = \frac{k_2(k_P + k_D s)}{s(RJs + RB + k_1k_2)} = D(s)$$

$$D_c(s) = \frac{k_2(k_P + k_D s)}{s(RJs + RB + k_1k_2) + k_2(k_P + k_D s)}$$

$$= \frac{k_2k_P + k_2k_D s}{RJs^2 + RBs + k_1k_2 s + k_2k_P + k_Dk_2 s}$$

$$= \frac{k_2k_D s + k_2k_P}{RJ\,s^2 + (RB + k_1k_2 + k_Dk_2)s + k_2k_P}$$

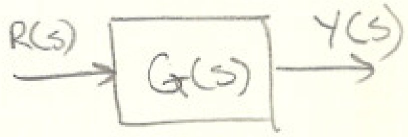
$$= \frac{\frac{k_2k_D}{RJ}\left(s + \frac{\cancel{k_2}k_P}{\cancel{k_2}k_D}\right)}{s^2 + \left(\frac{RB + k_1k_2 + k_2k_D}{RJ}\right)s + \frac{k_2k_P}{RJ}}$$

compare denominator to $s^2 + 2\zeta\omega_n s + \omega_n^2$

from T_s and P.O. we find ζ and ω_n from comparing.

STEADY STATE ERROR

open loop system



$$E(s) = R(s) - Y(s) = R(s) - R(s)G(s) = R(s)[1 - G(s)]$$

if $G(s)$ is stable.

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\,R(s)[1 - G(s)]$$

if $R(s) = \frac{1}{s}$

$$\lim_{t\to\infty} e(t) = 1 - G(0)$$

$$G_c(s) = \frac{G(s)}{1 + G(s)}$$

$$E(s) = R(s) - Y(s) = R(s) - \frac{R(s)G(s)}{1 + G(s)}$$

$$= R(s)\left[\frac{1 + G(s) - G(s)}{1 + G(s)}\right] = \frac{R(s)}{1 + G(s)}$$

if $G(s)$ is stable

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)\frac{1}{1+G(s)}$$

if, $R(s) = \frac{1}{s}$

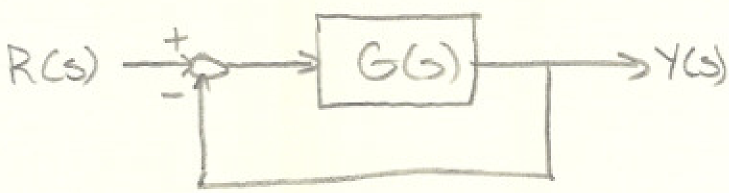
$$\lim_{t\to\infty} e(t) = \frac{1}{1+G(0)}$$

EX:

$$\frac{s+1000}{s^2+3s+1} \quad \text{(large DC gain)}$$

you can see that open loop error is large. however, closed loop will have a very small steady state error.

TYPE NUMBER OF SYSTEMS AND STEADY STATE.



$$sse = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

let $$G(s) = \frac{k\prod_{i=1}^{m}(s+z_i)}{s^n \prod_{k=1}^{Q}(s+P_k)}$$

the number of integrations N, is called the TYPE NUMBER of the system.

step input for a type number of $\emptyset$ ($N=\emptyset$) the sse for a step input of magnitude A, is:

$$sse = \lim_{s\to 0} \frac{sR(s)A}{1+G(s)} = \frac{A}{G(0)+1}$$

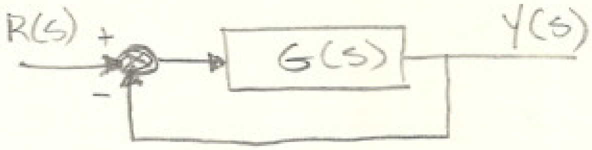
$$= \lim_{s\to 0} \frac{A}{1 + \boxed{\dfrac{k\prod_{i=1}^{m}(s+z_i)}{\prod_{k=1}^{\emptyset}(s+P_k)}}} = 0$$

goes to ∞

$G(0)$ (position error constant)

if $N \geq 1$, the steady state error is equal to zero.

RAMP INPUT.



$$G(s) = \frac{K \prod_{i=1}^{m} (s + z_i)}{s^n \prod_{k=1}^{Q} (s + P_k)}$$

$$R(s) = \frac{A}{s^2} \qquad r(t) = At$$

$$sse = \lim_{s \to 0} \frac{s(A/s^2)}{1 + G(s)} = \lim_{s \to 0} \frac{A}{s + sG(s)} = \lim_{s \to 0} \frac{A}{sG(s)}$$

* for a type zero ($N=0$) the steady state error is infinity.
* for a type one ($N=1$), the steady state error

$$\lim_{s \to 0} \frac{A}{sG(s)} = \frac{A}{\dfrac{K \prod_{i=1}^{m} z_i}{\prod_{k=1}^{Q} P_i}}$$

* if $N \geq 2$, then sse is zero.